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ANNUITIES AND ACTUARIAL SCIENCE PRINCIPLES AND DEFINITIONS

BY

R. J. Bennett, C. A., C. P. A.

Lesson 16

BENNETT ACCOUNTANCY INSTITUTE
PHILADELPHIA, PA.

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ANNUITIES AND ACTUARIAL SCIENCE

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R. J. Bennett, C. P. A.

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LESSON 16

ANNUITIES AND ACTUARIAL SCIENCE

PART I—PRINCIPLES AND DEFINITIONS

This lesson deals briefly with compound interest and its application. It is not exhaustive, since that would require a great deal of space and the inclusion of difficult mathematical calculations. Only the elementary actuarial principles are covered and such annuity problems as are considered of interest to the accountant. The subject is made as simple as possible, but even at that considerable study is required to master its various phases. At best annuity calculations are not particularly easy, but they can be readily understood by any person who has a thorough grasp of compound interest.

Actuarial science has to do largely with insurance probabilities and the application thereto of compound interest in the most intricate form, while the accountant is concerned only with the elementary principles thereof as set forth herein.

How to Study Annuities

In all of the annuity calculations the simplest method is followed and the compound interest table used as a basis. Arithmetical principles are used throughout. Algebraic principles are also interwoven as a more convenient means of presenting illustrations, but not that this is compulsory. Any one who has forgotten his knowledge of algebra however or who has never studied the subject should endeavor to familiarize himself with at least the principles thereof.

For an exhaustive study of the subject of annuities, the reader is referred to "The Accountancy of Investment" by Sprague, to which splendid book frequent reference has been made during the preparation of this lesson.

This lesson includes simple interest, discount, present worth or present value, true discount, compound interest, equations, etc. The tables used for annuity calculations include the compound interest table which may be found in any high-school arithmetic, the annuity tables, tables of logarithms and the bond tables. For a more detailed review of interest and discount, the reader is referred to any good high-school or commercial arithmetic.

Examination Requirements

Annuity problems under the title of "Elementary Actuarial Science" form a compulsory part of the examinations of the American Institute of Accountants. They are frequently to be found also in C. P. A. examinations of different states as well as in the Canadian examinations for Chartered Accountant. The author of this lesson believes that he extended some influence toward the inclusion of annuities in accountancy examinations. He was required to familiarize himself with them for the "C. A." examinations of the Institute of Chartered Accountants of Ontario, the final examination of which he passed in 1902. Later he used them quite extensively in his class lectures on accountancy, made them a necessary part of his accountancy courses, and used them in a number of articles contributed to business and accountancy magazines. In

1915, he read a paper on Annuities before the Pennsylvania Institute of Certified Public Accountants, in which were included some of the illustrations contained in this lesson. He believes therefore that he has done his part in disseminating information on this subject and in creating a desire for the study of an interesting division of the science of mathematics.

INTEREST AND DISCOUNT

In almost every commercial activity the business man or accountant has to do with interest in some form or other, either as an addition in the form of simple or compound interest or as a deduction in the form of discount. While the principles underlying interest and discount are familiar to all persons concerned in the study of annuities, it is thought advisable to include a brief review thereof before passing on to the more difficult application of compound interest to problems involving annuities.

Definitions

Interest is the amount paid or allowed for the use of money, or "money paid for the use of money," or "the increase of indebtedness through lapse of time." It may be either simple or compound according to agreement, though in calculations affecting investments, annuities and life insurance premiums compound interest is always understood. Rent and interest are practically the same, both being charged according to the time used.

Simple Interest is interest on the principal only, being the kind concerned in most business transactions. Interest is supposed to be paid when due, but if not it increases the amount owing though interest thereafter can be charged only on the original principal. No matter how much interest is in arrears, only the original amount is permitted to draw interest.

Compound Interest is interest on the principal and on the interest after it becomes due. This is the kind concerned in all problems affecting bond transactions, annuities and actuarial calculations. Indeed compound interest is the only kind that is adaptable to scientific problems.

Accurate Interest means interest determined on the basis of 365 days to a year, on the principle that the interest for say 25 days would be 25/365 of the full amount for the year.

Ordinary Interest or the Six Percent Method is formed on a basis of 360 days to a year or 30 days to a month. These terms have reference of course only to commercial transactions and not necessarily to scientific calculations.

Principal means the amount of the original investment or loan. In practice and in all calculations $1 is considered as the base or principal, after which the interest thereon or the final amount including interest is multiplied by the given number of dollars to obtain the desired result.

Investment means the amount invested or loaned for the sake of profit. This usually comprehends the sum invested in securities or in other ways apart and distinct from one's regular business or profession. The amount invested is the principal referred to in the preceding paragraph.

Discount is the sum deducted from a debt for payment before maturity, or the allowance made from a sum of money for prompt payment, or for the advancement of a sum of money.

Trade Discount is the percentage deducted from the face of a bill or list price of goods, being expressed by the word percent. without

regard to time. It is a term used entirely in commercial transactions, the rates of discount varying in each case according to the trade or custom prevailing in the particular industry or according to the fluctuations of the market.

Cash Discount has reference to an allowance made for prompt cash payment, being also expressed by the term percent. For example, a discount of 2% for payment within 10 days is a common business transaction.

Bank Discount has reference to the discount deducted by a bank before making a loan, or an amount allowed for the payment of a debt before it is due, the discount being reckoned according to the time of anticipated payment. The discount is determined by taking the simple interest on the amount of the debt from the day it is discounted to the date of maturity.

True Discount is the difference between any sum of money payable at a future time and its present worth, being equal to the interest on the present worth. The present worth of a debt or obligation payable at a future time without interest is such sum as being placed at interest at the legal rate will equal the given debt when it becomes due. "Present worth and true discount" are familiar terms to all students of arithmetic.

We see that $1,000 placed at 6% simple interest for 2 years will amount to $1,120, in which the $1,000 is the present worth of $1,120. The latter may be called the future worth.

A company discounts its 90 day note for $10,000 at the bank at a time when money is worth 6% in which case the bank deducts $150 for interest and credits the company with $9,850. Even though the full $10,000 must be paid at maturity, it is manifest that the company received only $9,850 for the use of which it pays an interest charge of $150. While the nominal rate based on $10,000 is 6% we see that the true or "effective" rate based on $9,850 is 6.09 plus %. The bank discount therefore is seen to be 6% while the true discount is in excess of that rate.

PART II—COMPOUND INTEREST

The mastery of actuarial and annuity calculations depends entirely upon the knowledge and application of compound interest. Interest, discount, present worth, future worth, final value, principal, investment, rate, time, period, powers, roots, equations, formulas, are terms that obviously must be understood in solving problems of this nature.

Use of Compound Interest

Compound interest is used in all calculations affecting annuities on the theory that money keeps earning money and that the earnings in turn are immediately placed at interest and made to earn more money. The interest accretions therefore become in turn a part of the principal, which in each succeeding interest period forms the new principal. The rapidity with which the investment increases must obviously depend upon the rate of interest and upon the frequency of the interest periods. Given the same rate of interest, it is manifest that the accumulations will be greater if compounded semi-annually than annually, or more rapid quarterly than semi-annually.

For example, an investment of $1.00 for one year at 5% will at the end thereof amount to $1.05; if compounded semi-annually (two interest periods at 2½%) it will amount to $1.050625; and if compounded

quarterly (four periods at 1¼%), it will amount to $1.05094534. It is manifest that the frequency with which the interest accretions are converted into principal governs the ultimate increase of the investment. Therefore the nominal rate per annum in this illustration is 5% while the effective rate is 5.0625% and 5.094534% respectively, which based on an investment of $1,000 would amount at the end of one year to $1,050, $1,050.62 and $1,050.95. When stating the rate percent., it is also necessary to designate the interest period, as annually, semi-annually, quarterly or monthly as the case may be; as 6% per annum compounded quarterly, 5% per annum compounded half yearly, and so forth.

Theory of Increase

The frequency of the interest periods governs very largely the rapidity with which the investment increases as the interest instalments are gradually added thereto and in turn become principal. The conversion of interest into principal and in turn its creation of more interest, lead us to a further study of the terms applied to the entire process.

In computing compound interest, 1 is usually taken as the basis or as the amount of investment since the result thereof can be so readily adapted to any amount of larger dimensions.

If, for example, $1.00 placed at interest for 6 years and compounded annually at 5% amounts to $1.3400956, we see that it is the result of the regular annual addition of 5 cents for 6 years together with the accretions of interest on these interest instalments. During the life of the investment the 5 cents are added six times and become in reality an annuity of 5 cents for 6 years.

Suppose for example the original $1.00 to have been left in one bank for the entire period of 6 years and the annual interest of 5 cents withdrawn each time as soon as it became due and deposited at the same rate of interest in a savings bank, it is clear that at the end of 6 years we shall have just the original $1.00 in the first named bank and 34 cents in the other, or to be exact .3400956 cents. This amount in the savings bank is simply the result of an annuity of 5 cents per annum deposited therein for a period of 6 years. Therefore it will be seen that the compound interest is made up of annuities or regular annual accretions of given amounts, and that the compound interest table itself forms the basis of the annuity table. The interest instalments are the annuities. From the compound interest tables can readily be obtained the final value of any annuity desired, as will be shown in succeeding examples and from them a set of annuity tables could easily be formulated.

At the end of the first year the investment will have amounted to $1.05; at the end of the second year to $1.05 multiplied by itself (1.05x 1.05) or $1.1025; at the end of the third year to $1.1025 x 1.05 or 1.157625, and so on until the end of the sixth year which results in a total of $1.3400956. It will be seen that this amount is obtained by multiplying the ratio of increase (1.05) as many times as there are periods. In this case the $1.05 would be raised to the 6th power—that is, multiplied by itself 6 times, otherwise expressed $(1.05)^6$. The (1.05) is termed the *ratio of increase.*

By combining the ratio of increase and powers the above results may be expressed in symbols as follows:

$(1.05)^2$ gives 1.1025; $(1.05)^3$ gives $1.157625, and so on, $(1.05)^6$ giving a total of $1.34 as shown above.

Methods of Solution

It is manifest that the greater part of the work consists in solving $(1.05)^6$, which may be done in any one of three ways.

(a) By actual multiplying
(b) By reference to the compound interest table
(c) By logarithms

By actual calculation $(1.05)^6$ is determined by multiplying the ratio of increase as many times as there are periods, 6 times, as follows:

Times	Products
	1.00 is the investment
(1)	1.05 is the ratio of increase or multiplier
	1.05 is the amount at end of 1 year
(2)	1.05
	1.1025 is the amount at end of 2 years
(3)	1.05
	1.157625 is the amount at end of 3 years
(4)	1.05
	1.2155063 is the amount at end of 4 years
(5)	1.05
	1.2762816 is the amount at end of 5 years
(6)	1.05
	1.3400956 is the amount at end of 6 years

Considerable work is involved of course in working out all of these multiplications and therefore the compound interest table is resorted to since it contains the given results for various periods and at different rates of interest.

There is a short cut however to this long method of calculation, by making use of two of the powers concerned and multiplying them together, providing these two are represented by an exponent equal to the sum of two exponents of the power multiplied. There are for example six powers in the above illustration from which we can obtain groupings of 3 and 3, or 2 and 4 either of which will produce the given result. We see that

(a) $1.05^6 = 1.3400956$
(b) $1.05^3 \times 1.05^3 = 1.3400956$
(c) $1.05^2 \times 1.05^4 = 1.3400956$

In the above expression $(1.05)^6$, the 6 is the exponent and the result, the power—as 1.3400956 is the resultant 6th power.

By going a step further and using the amounts obtained above by interpolation, we get the same result, as follows:

(a) 1.05 for 6 times = $1.3400956
(b) 1.157625 × 1.157625 = 1.3400956
(c) 1.1025 × 1.2155063 = 1.3400956

In like manner the amount for any number of years not given in the compound interest table may be computed by finding the products of any two numbers of years whose sum equals the given time. For example, the compound amount for 10 years multiplied by the compound amount for 20 years will give the total compound amount for 30 years. This method is termed *interpolation.*

Logarithms

By means of logarithms the computation can be very much reduced, but since that requires the use of a table of logarithms, no attempt is made to illustrate their use. Such tables can be found however in almost any good algebra or in Sprague's "Accountancy of Investment" already referred to.

PROBLEMS IN COMPOUND INTEREST

Example 1.—What sum will $500 amount to in 6 years at 5% compounded annually?

Answer.—We find from the compound interest table that 1.05^6 amounts to $1.3400956 therefore $500 \times 1.3400956 =$ the answer, $670.047800 or $670.05. This can also be determined by continuous multiplying as already illustrated.

Example 2.—Find the compound interest on $500 for 6 years at 5%.

Answer.—The compound amount less the original investment gives the compound interest. In the above example, 1.3400956 is the compound amount including principal and interest, while .3400956 is the interest: this .3400956 multiplied by 500 gives the answer, $170.05, which also may be obtained by subtracting 500 from the total shown in example one, $670.05—$500 = $170.05.

Example 3.—What sum will $500 amount to in 3 years at 6% compounded semi-annually?

Answer.—This is equivalent to 6 years at 3%.

$1.03^6 = 1.1940523$

$500 \times 1.1940523 =$ answer, 597.02615

Then $500 will amount to $597.03

The interest alone is $97.03.

Example 4.—Find the compound interest on $2,000 for 2 years and 3 months compounded annually at 8%.

Answer.—The answer is $379.45, determined as follows:

Find the compound interest for the 2 years and then for the 3 months.

Or, multiply the successive ratios $1.08 \times 1.08 \times 1.02 \times \$2000 = \$2,379.456—\$2000 = \$379.45$.

Or, $1.08^2 \times 1.02 \times 2000 =$ the total amount $2,379.456

Then $2,379.45—$2000 = $379.45 or the amount of interest.

Example 5.—What is the present worth of $4,000 due in 3 years at 6% compound interest, compounded half yearly?

Answer.—First determine the final value of $1.00 for 6 periods or years at 3%, which is $1.1940523.

We see that $1.00 is the present worth of $1.19, or that the final value or future worth is practically 119% of the present worth.

Then, if $1.1940523 has a present worth of $1.00, $4,000 must have a present worth of $4,000 \div 1.1940523$, or $3,349.93704.

Then $\frac{4.000}{1.03^6} = \$3,349.94$.

Rule.—To find the present worth of an amount at compound interest, divide the given amount by the amount of $1 for the given time and rate at compound interest.

Example 6.—If money is worth 7% compounded half yearly which is the better offer—$5,000 cash down or $6,000 at the end of 3 years without interest?

Answer.—$1 due in 6 years at 3½% has a present value worth of $\frac{1.00}{1.035^6} = .81350064$

Then $6,000 has a present worth of $6,000 \times .81350064 = \$4,881.00384$

Or, $6,000 \div 1.035^6 = \frac{6,000}{1.22925533} = \$4,881.00$

The offer of $5,000 cash down is better by $119. $5,000—$4,881 = $119.

Example 7.—What principal placed at compound interest at 6% compounded semi-annually will amount to $10,000 in 4 years?

Answer.—Find the final value of $1 at compound interest for 8 periods at 3% and divide into the given amount. This will give the required answer.

Then $1.03^8 = 1.26677008$

Therefore $\$10,000 \div 1.26677008 =$ the answer, $7,894.09.

Or, from the "present value" table take .78940923, the present worth of $1, and multiply it by 10,000 which will equal $7,894.0923.

Example 8.—At what rate of interest compounded annually would $4,000 have to be loaned to amount to $5,105.13 in 5 years?

Answer.—The final value given is seen to be $5,105.13 while the difference, or $1,105.13, is the compound interest. Without the aid of the interest tables however the problem is not easy of solution.

If $4,000 will yield $1,105.13 interest in 5 years, $1 during the same time will yield $\frac{1}{4000}$ of $1,105.13, or .2762825; then $1 will amount in 5 years to $1.2762825. In the compound interest table opposite 5 years we find in the 5% column the amount of $1 to be $1.27628156 which is practically the amount required. Therefore the rate is 5%.

PART III—ANNUITIES

PRINCIPLES AND DEFINITIONS

The preceding explanations respecting interest were only preliminary to the subject now to be treated. In solving annuity problems however the principles already established can be used to advantage. There is hardly a problem therein but that can be solved by a careful manipulation of the compound interest tables. The *amount* of the annuity and the *present value* are chief considerations in computations of this kind, as will be seen in the following illustrations.

Annuities are constantly being used in connection with investments of all kinds, bond issues, insurance premiums, mortgage payments, sinking funds, rent instalments, depreciation, amortization, and so forth. For annuity calculations the tables may be used, but notwithstanding this convenience, the accountant should be able to reason the questions out for himself.

Definitions

Before beginning the discussion of annuities, let us familiarize ourselves with a few of the terms peculiar to annuity calculations.

An annuity is a payment of a definite sum of money annually or at regularly recurring intervals, as half-yearly, quarterly, etc.

An annuity certain, or certain annuity, is one which begins and ends at fixed times.

A perpetual annuity, or perpetuity, is one which continues permanently, forever.

An annuity in possession, or immediate annuity, is one that begins immediately.

A deferred annuity, or annuity in reversion, is one that begins at some future time which may be at a specified date or at the occasion of some event.

Annuity in reversion, or a deferred annuity, is one that begins at some future time which may be at some specified time or at the occasion of some event.

An annuity in arrears, or foreborne, is one on which payments were not made when due.

The amount of an annuity means the total sum to which the periodical instalments plus compound interest thereon will amount at the end of a given time.

The present value, or present worth of an annuity, is the amount which placed at compound interest will equal the final value of an annuity at the end of a given number of years at a given rate. It is the sum which placed at interest would produce the desired annuities.

Amortization means the gradual reduction or extinguishing of a debt or amount. It is very similar to depreciation.

Sinking Fund consists of amounts set aside at regular intervals and accumulated at compound interest for the purpose of meeting some debt, mortgage or bond issue when it becomes due. The annuity instalments plus the accumulation of interest are presumed to be sufficient to equal the debt at maturity.

We see from the above definitions that an annuity is a payment of a definite sum of money annually or at regularly recurring intervals. The payments, periods and rates of interest being uniform, the calculation of the amounts is much simplified, but where a number of variables creep in, the problem becomes complicated. The annuities or periodical payments are known also as *rents.*

DETERMINING THE AMOUNT OF AN ANNUITY

The illustrations which follow pertain to annuities, exemplifying with graded problems the methods of solution. The student is advised however to provide his own solutions before consulting the answers.

At this stage a review of the instructions pertaining to compound interest as set forth in the preceding pages is most advisable. Examples 1 to 8 should be thoroughly mastered before beginning this section of thework.

Explanation and Rule

By referring to the preceding explanations, we find that \$1 compounded annually at 5% for 6 years amounted to \$1 plus .34009564, or \$1.34009564. This amount may be obtained from the compound interest table or computed according to the following formula:

$\$1.05^6$	= \$1.34009564
The principal invested =	1.00
The interest earned =	.34009564
The compound amount =	1.34009564

From the above we see that .34009564 is the value at maturity of a 5 cent annuity—that is, the final value of an annuity of 5 cents for 6 years at 5%. The rate is expressed as .05. Therefore from this result the final value of an annuity of 1 cent, 1 dollar, or any other sum may be readily found. To find the value of an annuity of 1 cent, divide the above result by 5, and to get the value of 100 cents, or \$1.00, multiply the result of 1 cent by 100, as per the following formula:

Final value of an annuity of 5 cents for 6 years = .34009564

Final value of an annuity of 1 cent for 6 years $= \frac{.34009564}{5}$

Final value of an annuity of 100 cents for 6 years $= \frac{.34009564}{5} \times 100 =$ 6.8019128

To simplify the above expression omit the 100 and use .05 for the rate,

Thus $\frac{.34009564}{.05} = 6.8019128$

Therefore an annuity of \$1.00 for 6 years at 5 percent. amounts to \$6.8019128, which can be verified by reference to the annuity table. The final value of an annuity of any given amount for the same time and rate can be easily determined by multiplying it by this result. The process is simple and may be set forth together with other plans in the following rules.

Rule.—To find the final value of an annuity of \$1.00 for a given time at a given rate, use any of the following three methods:

(1) Refer to the annuity table in the given rate column on the line corresponding to the given number of years.

(2) Find the compound interest on \$1.00 for the given time at the given rate from the compound interest table, or by calculation, then multiply this result by 100 and divide by the given rate.

(3) Determine by calculation or from the compound interest tables the final value separately of each given annuity of \$1.00 for the given number of years, and add the results together—on this basis the above would be expressed by determining and adding together the compound amount of \$1.00 for 5, 4, 3, 2, 1 and 0 years.

Methods of Calculation

The manner of determining the final value of a given annuity by each of the three different methods suggested above will now be illustrated, as follows:

Example 9.—What is the final value of an annuity of \$500 for 4 years at 5%, interest compounded annually?

Answer.—Compound interest is used on the assumption that money is re-invested immediately at each interest date. In this example the interest is compounded annually. We see that \$500 is to be put out at interest each year at 5%. The instalments therefore will amount to \$2,000, and on payment of the final instalment the desired end will have been accomplished so that the final payment will not draw interest. The annuity payments will draw interest as follows:

Instalment No. 1 will draw compound interest for 3 years.
Instalment No. 2 will draw compound interest for 2 years.
Instalment No. 3 will draw compound interest for 1 year.
Instalment No. 4 will draw compound interest for 0 years.

Plan 1.—Solution from annuity table:

Amount of annuity of \$1 for 4 years at 5% = \$4.310125.

Amount of annuity of \$500 for 4 years at 5% = \$4.310125 × 500 = \$2,155.0625.

The resultant answer therefore is $2,155.06.

Plan 2.—Solution from compound interest table:

Amount of $1 at compound interest for 4 years at 5% = $1.21550625

The compound interest for 4 years at 5% = .21550625

Then $\frac{.21550625}{.05} = \$4.310125$, the final value of $1 annuity.

Therefore $4.310125 × 500 = $2,155.06, the answer.

Or, $\frac{(1 \text{ plus } 5)^4—1}{5} \times 100 = \$4.310125$ which × 500 = answer.

Plan 3.—Solution by calculation:

(a) Formula to determine value of annuity of $1.00:

Amount of $1 compounded for 3 years $(1.05)^3 = \$1.157625$
Amount of $1 compounded for 2 years $(1.05)^2 =$ 1.102500
Amount of $1 compounded for 1 year $(1.05)^1 =$ 1.050000
Amount of $1 compounded for 0 years (1.00) = 1.000000
Amount of $1 annuity for 4 years = $4.310125

Then $4.310125 × 500 = $2,155.06, the answer.

(b) The calculations may be made clearer by a further analysis of each instalment and its accumulation of interest, as follows:

		Annuity	**Interest**	**Final Amount**
1	First annuity payment.........	$ 500	....	....
	Compound interest on same for 3 years............................		$ 78.81	$ 578.81
2	Second annuity payment.......	500	....	....
	Compound interest on same for 2 years...........................		51.25	551.25
3	Third annuity payment........	500	....	....
	Compound interest on same for 1 year............................		25.00	525.00
4	Fourth annuity payment.......	500	....	....
	No interest........................	.....	....	500.00
	Answer—Final value of all annuities.................	$2,000	$155.06	$2,155.06

Algebraic Formula

The algebraic formulas simply make use of letters or symbols in place of figures as a convenient means of expression but afterward they are converted into figures as will be seen below.

Giving expression to the matter in algebraic terms we have the following formula:

Let r equal the rate of interest
Let n equal the time or number or periods
Let I equal the annuity of $1
Let (1 & r) equal the ratio of increase
The terms "plus" and "&" are synonymous

Then $\frac{(I \text{ plus } r)^n—1}{r}$ = the amount of $1 annuity, giving the following equation: $\frac{(I \text{ plus } 5)^4—1}{.05} = \frac{.21550625}{.05} = \$4.310125$, amount of $1 annuity

Then $4.310125 × 500 = $2,155.06, the answer.

The symbolic formula $\frac{(1 \text{ plus } r)^n—1}{r}$ is analyzed and simplified in the following application.

$500 \times \frac{(1.05)^4—1}{.05} = 500 \times \frac{(1.21550625)—1}{.05} = 500 \times \frac{(.21550625)}{.05} =$ $2,155.06.

The greater part of the work consists of course in solving $(1.05)^4$, which as already explained may be simplified by the use of logarithms.

Wentworth's Complete Algebra, page 377, illustrates the solution of a similar problem by the same formula.

SUNDRY PROBLEMS IN ANNUITIES

For further elucidation of this principle the formulas are applied to the solution of other examples of a similar nature. These should be studied with care.

Example 10.—A man pays $150 yearly for 15 years for an endowment policy of $2,500. Find the accumulated value of payments if money is worth 6 per cent.

Answer.—$150 \times \frac{(1.06)^{15}—1}{.06} = \$3,491.40$

This answer can be proved by reference to the annuity tables, or obtained from the compound interest table, as follows:

The table shows $(1.06)^{15}$ to be 2.39655819, the amount of $1 compounded.

Then 2.39655819—1 = 1.39655819 or the final value of an annuity of 6 cents.

Then $\frac{1.39655819}{.06} = 23.27596988$, the amount of an annuity of $1.

Then 150 × 23.27596988 = $3,491.395, the answer.

Example 11.—Find the amount of an annuity of $400 for 3 years, reckoning interest at 4 per cent. compounded semi-annually.

Answer.—In this example we see that the annuities are made annually while the interest thereon is reckoned semi-annually, therefore the effective annual rate is used in calculating. The 4% compounded semi-annually is equal to an effective annual rate of 4.04%.

Thus $(1.02)^2 = 1.0404—1 = .0404\%$ for one year

Then $(1.02)^2—1 = 4.04\%$ the effective rate.

Then $400 \times \frac{(1.0404)^3—1}{.0404} = \$1,249.13$, the answer.

The result of $(1.02)^6$ is the same as $(1.0404)^3 = 1.12616242$.

Example 12.—Find the amount accumulated at the end of 10 years by a person who deposits in a bank at the beginning of each year the sum of $200, the bank paying 4% interest, compounded yearly.

Answer.—Since the deposits are made at the beginning of the year instead of at the end as in ordinary annuities, the first deposit bears interest for 10 years—not 9 years as in the case of a 10-year annuity. This is called an *annuity due* or prepaid to distinguish it from the ordinary. The answer may therefore be obtained in two ways, as follows:

(a) Find the solution to a regular 10-year annuity, and to the result add the compound interest on $1 for 10 years which is left out in the reckoning of the annuity as ordinarily done.

(b) Or, from the amount of an ordinary 11-year annuity deduct \$1, which result will be the same as that of a 10-year annuity with interest starting at the beginning of the year instead of at the end.

The solutions under these two plans are set forth below.

Plan (a).— $200\left(\frac{(1.04)^{10}-1}{.04}+(1.04)^{10}-1\right)=\$2,497\ 27$

An analysis of the formula gives the following equations:

$\frac{(1.04)^{10}-1}{.04}=\frac{1.48024428-1}{.04}=\$12.006107$ = amount of \$1 annuity,

$(1.04)^{10}-1=.48024428$ = compound interest for 10 years.

Then 12.006107 plus .48024428 = 12.48635128, or the final amount of \$1 annuity due.

Therefore \$200 × 12.48635128 = \$2,497.27, the answer.

Plan (b).— $\frac{(1.04)^{11}-1}{.04}=13.48635141,$

And 13.48635141—1 = 12.48635141,

Then \$200 × 12.48635141 = \$2,497.27, the answer.

PART IV—SINKING FUNDS

DETERMINING ANNUITY WHEN AMOUNT, TIME, AND RATE ARE GIVEN

We have already illustrated the application of compound interest to annuity calculations and the manner of determining results as applied to simple problems.

Given the annual payment, the time, and the rate, we have found the amount or final value of an annuity. Let us now go a step further, but still making use of the formula already learned for deducing the rule for our second line of problems. This section takes up the manner of determining the annual payments required to satisfy a given debt when the time and rate are given. This principle is particularly serviceable in arriving at the amount of sinking fund instalments required to meet a given bond issue, mortgage, lease, or other obligation. The final amount due or to be accumulated is called the *sinking fund*, though the periodical instalments are frequently spoken of as the sinking fund. The latter are known as the sinking fund instalments or contributions.

Formula and Rule

By referring back to our basic formula we see that the

Amount or final value of an annuity = Annuity $\times\frac{(1\ \&\ r)^n-1}{r}$

If by this process the final value of several annuities can be determined it is manifest that by reversing the process it is possible to work backward from the amount to the annuity payments themselves. To find the annuity then proceed as follows:

Amount $\div\frac{(1\ \&\ r)^n-1}{r}$ = the annuity. Since in dividing by a fraction the divisor is inverted, we then have the following:

Amount $\times\frac{r}{(1\ \&\ r)^n-1}=\frac{Ar}{(1\ \&\ r)^n-1}$ = the annuity

It will be seen that by the formula the final amount is divided by the

result of an annuity of one dollar. This principle may be expressed in the following rule:

Rule.—Divide the amount due at maturity by the amount of an annuity of \$1 for the given time at the given rate, and the quotient will be the annuity required.

The manner of handling this class of problems is exemplified in the following illustrations.

PROBLEMS IN SINKING FUNDS

Example 13.—What sum of money deposited at the end of each year for the next ten years will amount to \$8,000, money being worth 5% per annum?

Answer.—The answer according to the above formula is briefly stated, as follows:

$8,000\div\left(\frac{(1.05)^{10}-1}{.05}\right)=8,000\times\left(\frac{.05}{(1.05)^{10}-1}\right)=\$636.04$, the annuity

Or, $\frac{8000\times.05}{(1.05)^{10}-1}=\frac{400}{1.62889463-1}=\frac{400}{.62889463}=\$636.0366$, the answer.

The principles involved are fully analyzed under the following example.

Example 14.—On January 1, 1920, a railroad company issued \$5,000,000 of 5% sinking fund bonds payable in twenty years, interest coupons payable semi-annually. Assuming that the annual instalments must be equal in amount and that they will draw interest compounded at 4%, what sum must be set aside at the end of each year to meet the debt?

Answer.—It will be seen that \$250,000 must be raised each year, \$125,000 half yearly, for coupon payments in addition to the sinking fund instalment, but the interest payments will not affect the annuity instalments in any way. As the interest coupons mature, they are paid and the amount charged to Bond Interest account, which in turn is closed into Profit and Loss.

In twenty years the railroad company will have to pay \$5,000,000, and during that time it must set aside at the end of each year a predetermined definite sum of money to remain at compound interest until the maturity of the bonds. It will be readily seen that the first instalment or annuity will draw interest at 4% compounded for 19 years, the second for 18 years, the third for 17 years, and so on until the last instalment is paid, which will be at the date of maturity of the bonds and therefore will not draw any interest.

In determining the required annuities, \$1 must be taken as the base; then determine the final value of an annuity of \$1 for 20 years at 4 per cent, and divide this result into the \$5,000,000. The answer will result. This can be obtained in any one of three ways, as follows:

(1) By consulting the annuity tables.
(2) By carrying out the formula described in the above rule.
(3) By ordinary calculation.

Each of these plans of solving the question will now be illustrated.

Plan 1.(a) From the "amount of an annuity" table get the final value of an annuity for the 20 years at 4%, which is seen to be \$29.77807858; this amount divided into \$5,000,000 gives the required answer.

Thus $\frac{\$5,000,000}{\$29.77807858}$ = \$167,908.75 the answer.

If the \$29.77807858 is the result of a known annuity of \$1, then \$5,000,000 is the result of an annuity which is as many times greater than \$1 as \$29.77807858 is contained in \$5,000,000.

(b) From the "sinking fund table" it can also be obtained. This table shows what amount deposited at the end of each year will amount to \$1. In this case it is .03358175, determined thus:

$$\frac{\$1.00}{29.77807858} = .03358175$$

Then \$5,000,000 × .03358175 = \$167,908.75, answer.

Plan 2.—Under this plan determine the amount of \$1 at compound interest for 20 years at 4%, which is found to be \$2.19112314; then deduct the \$1 principal and divide the remainder by .04 obtaining the result of an annuity of \$1 for 20 years at 4%, as follows:

\$2.19112314—\$1 = \$1.19112314 ÷ .04 = \$29.77807858

Or, $\frac{(1.04)^{20}-1}{.04}$ = \$29.77807858, final value of \$1.

Then $\frac{\$5,000,000}{29.77807858}$ = \$167,908.75, the required instalment.

This amount set aside at the end of each year will equal the maturing debt and can be proved by actual calculation.

Plan 3.—Separately compute the final value of \$1 at compound interest for from 1 to 19 years; these added together with the final annuity which draws no interest at all will give the final value. This gives \$29.77807858, from which we proceed as shown in the two preceding plans.

Condition of Sinking Fund

Certain bookkeeping requirements are necessary in connection with a sinking fund of this kind to show the annual instalments, the interest accumulations, and the condition of the fund at any time. The table shown below provides the desired information.

The first instalment to the sinking fund will immediately begin accumulating interest, and the condition of the fund from year to year thereafter will be as follows:

SINKING FUND ACCOUNT

End of Year	Annuity	4% Earnings	Amount in Fund
1920	\$167,908.75	....	\$167,908.75
1921	167,908.75	\$6,716.35	342,533.85
1922	167,908.75	13,701.35	524,143.95
1923	167,908.75	20,965.76	713,018.46
1924	167,908.75	28,520.74	909,447.95
1925	167,908.75	36,377.92	1,113,734.62
Add 1926—40	2,350,722.50	1,535,542.88	3,886,265.38
Total....	\$3,358,175.00	\$1,641,825.00	

SUMMARY

Total of 20 Instalments........................	\$3,358,175.00
Total interest earnings........................	1,641,825.00
Total debt............................	\$5,000,000.00

If the sinking fund is found at any time to be short of the amount required, it should either be increased by a separate deposit or the remaining instalments sufficiently increased to recover the shortage.

Example 15.—A manufacturing company in 1921 issued \$1,000,000, 6% sinking fund gold bonds with 10 years to run. The directors in planning for the sinking fund believed that during the first 5 years 5% interest could be earned by the trustee, and 4% thereafter. They want half the sum to be accumulated during the first five years and the remainder during the next five. What sum should be set aside each year to fulfill these stipulations?

Answer.—Variations in the sinking account are not unusual and it is business prudence to prepare for them in advance. In this problem there are two or three steps to be taken, as follows:

(a) Find the required annuity that will amount to \$500,000 in 5 years at 5%

Thus, $500,000\left(\frac{.05}{(1.05)^5-1}\right) = \frac{500,000}{5.52563125} = 90,487.40$

The annuity for the first 5 years is therefore \$90,487.40.

(b) Now then, the sinking fund at this stage has a total \$500,000, which amount will draw compound interest at 4% for 5 years regardless of the succeeding sinking fund instalments.

Thus, 500,000 × $(1.04)^5$ = \$608,326.45 at maturity.

(c) Therefore, since the sinking fund will already contain enough to give \$608,326.45 at maturity, only the difference between that amount and \$1,000,000 will have to be accumulated.

So that \$1,000,000—\$608,326.45 = \$391,673.55, the amount to be provided for during the second 5 years.

(d) Then $\frac{.04}{(1.04)^5-1} = 5.41632256$

So that $\frac{391,673.55}{5.41632256}$ = \$72,313.56, the instalments.

Therefore, the result may be summarized as follows:

Amount of \$90,487.40 instalments for 5 years =	\$ 500,000.00
Amount of interest on above for 5 years =	108,326.45
Amount of \$72,313.56 instalments for 5 years =	391,673.55
Total fund at maturity =	\$1,000,000.00

PART V—DETERMINING PRESENT VALUE OF AN ANNUITY

In the preceding examples we have seen how to obtain both final values and annuities. With these principles in mind we can readily solve the problems which follow, that of determining the present value of a given annuity providing the necessary data are given. It is manifest that the Present Worth in any case, whether covering annuities or otherwise, if multiplied by the compound amount of \$1 for the time should produce the final amount. That is to say, the present worth of

an annuity is the sum of money which placed at compound interest for the given time will equal the stated annuity payments plus the interest thereon.

The analysis thereof may be expressed in the following formula, in which "P. W." represents the present worth and "a" the annuity.

$$\text{P. W.} \times (1 \,\&\, r)^n = a\left(\frac{(1 \,\&\, r)^n - 1}{r}\right) \text{ or the amount of the annuity}$$

Now reverse the operation to get the present worth thereof

$$\text{Then } a\left(\frac{(1 \,\&\, r)^n - 1}{r}\right) \div (1 \,\&\, r)^n = \text{P. W.}$$

$$\text{Therefore, P. W.} = \frac{a}{r} \times \frac{(1 \,\&\, r)^n - 1}{(1 \,\&\, r)^n}$$

Rule.—To find the present worth of an annuity, determine the final value of the annuity for the given time at the given rate, and divide it by the compound amount of $1 for the given time at the given rate.

Example 15 (a).—A company agreed to pay $1,600 a year for five years. What sum paid now would be equivalent to this annuity, money being worth 4% per annum payable yearly?

Answer.—In answering this question first focus both principal and interest to a given date, the date of maturity of the last annuity, and then find the present worth and these two results. Since the principles have already been set forth we may proceed without further explanation. The problem may be solved in any one of the following ways.

(1) By reference to the annuity tables.
(2) By the formula given above, using the compound interest table as a basis.
(3) By actual calculation of each annuity separately.

These plans are illustrated briefly in the following examples:

Plan 1.—Multiply the annuity by the present worth of $1 annuity given in the 4% column of the "present worth" table showing the present worth of annuity for 5 years, which gives 4.45182233.

Thus $4.45182233 = Present worth of $1 annuity for 5 years at 4%.

Then $4.45182233 × $1,600 = $7,122.92, the answer.

Plan 2.—To get the present worth of $1 annuity, divide the final value of $1 annuity for 5 years at 4% by the amount of $1 compounded for the same time at the same rate, as follows:

Amount of $1 compounded for 5 years at 4% = $1.2166529.

$$\text{Amount of \$1 annuity for 5 years at } 4\% = \frac{.2166529}{.04} = 5.4163225.$$

$$\text{Then the Present Worth of \$1 annuity} = \frac{5.4163225}{1.2166529} = \$4.451822.$$

Then $4.451822 × 1,600 = $7,122.91, answer.

Or expressing another way by the following:

$$\text{Formula:—}\$1{,}600 \times \left(\frac{(1.04)^5 - 1}{.04}\right) \div (1.04)^5 = \$7{,}122.91.$$

Or, Final value of $1,600 annuity = $8,666.11

Present worth of $1.2166529 = $1.

$$\text{Present worth of } \$8{,}666.11 = \frac{8{,}666.11}{1.2166529} = \$7{,}122.91.$$

Plan 3.—We shall now illustrate by actual calculation the two methods of determining the present worth of each annuity separately.

(a) Separately find the Present Worth of each annuity of $1,600 and 4% interest on same for the time involved, as follows:

Amount of $1,600 for 0 years = $1,600.00 = 1600.
Amount of $1,600 for 1 year = 1,664.00 = $1600 \times (1.04)$.
Amount of $1,600 for 2 years = 1,730.56 = $1600 \times (1.04)^2$.
Amount of $1,600 for 3 years = 1,799.78 = $1600 \times (1.04)^3$.
Amount of $1,600 for 4 years = 1,871.77 = $1600 \times (1.04)^4$.
Amount of all annuities $8,666.11 $8,666.11

Now get the present worth of this amount due 5 years hence, which on the basis of $1 shows that the future worth is a little over 121% of the present worth—that is, 1.2166529.

$$\text{Then } \frac{8{,}666.11}{1.2166529} = \$7{,}122.91.$$

(b) Separately find the present worth of each annuity of $1,600 for the time involved, as follows:

$$\text{Present worth of \$1,600 due in 1 year} = \frac{1{,}600}{(1.04)} = \$1{,}538.46$$

$$\text{Present worth of \$1,600 due in 2 years} = \frac{1{,}600}{(1.04)^2} = 1{,}479.29$$

$$\text{Present worth of \$1,600 due in 3 years} = \frac{1{,}600}{(1.04)^3} = 1{,}422.39$$

$$\text{Present worth of \$1,600 due in 4 years} = \frac{1{,}600}{(1.04)^4} = 1{,}367.69$$

$$\text{Present worth of \$1,600 due in 5 years} = \frac{1{,}600}{(1.04)^5} = 1{,}315.08$$

Present worth of all annuities = $7,122.91

Proof of Answer

The above result can be proved by computing $7,122.91 for the full time and by comparing it with the result of the annuity of $1,600 for the given time. It is also shown in greater detail in the following table of reductions.

SUMMARY ANNUITY REDUCTIONS

Year	Beginning	Add Interest	Total	Deduct Annuity	Balance
1	$7,122.91	$284.91	$7,407.82	$1,600	$5,807.82
2	5,807.82	232.31	6,040.13	1,600	4,440.13
3	4,440.13	177.61	4,617.74	1,600	3,017.74
4	3,017.74	120.71	3,138.45	1,600	1,538.45
5	1,538.45	61.55	1,600.00	1,600	None
Total	....	$877.09	....	$8,000	....

Compound Discount is the difference between an amount due at a future date and its present worth. It is practically the compound interest on the present worth. For example, the present worth of $1 for 4 years at 6% is $\frac{1}{1.06^4}$ or .79209366, which can be obtained from a table of present values. Therefore the difference or .20790634 is the compound discount. From the compound interest table this can also be found by dividing the compound amount of $1 into the compound interest. Now then, $1.06^4 = 1.26247696$ which divided into $.26247696 = .20790634$.

Example 16.—What is the Present Value of an annuity of $2,500 for 15 years at 5% compound interest?

Answer.—Present worth $= 2{,}500\left(\frac{(1.05)^{15}-1}{.05}\right) \div (1.05)^{15}$

Then $\frac{(1.05)^{15}-1}{.05} = 21.57856359$, which $\times 2{,}500 = 53{,}946.41$.

And $(1.05)^{15} = 2.07892818$

Then $53,946.41 ÷ 2.07892818 = $25,949.15.

Example 17.—A suburban town borrowed $20,000 for street improvements and agreed to pay 5% compound interest on the debt for the entire time. What sum must be set aside each year as a sinking fund to pay the amount of principal and interest at the end of 12 years?

Answer.—Our first concern is to know how much will be due on the date of maturity comprising the $20,000 and compound interest thereon for 12 years. After this is obtained we simply continue as in the preceding example.

$20{,}000 \times (1.05)^{12} =$ the amount, $35,917.12

Then $35{,}917.12 \div \frac{(1.05)^{12}-1}{.05} = \$2{,}256.51$, the annuity.

The required answer therefore is $2,256.58 per annum.

Present Worth of Deferred Annuity

In a deferred annuity the instalments do not begin at once but are deferred for a stated number of years or periods. The following rule may therefore be used in cases of this kind.

Rule.—To find the present worth of a deferred annuity, determine the final value of the annuity for the number of years effective and divide the result by the compound amount of $1 for the entire time Or, multiply the final value of the annuity, as above, by the present worth of $1 for the entire time.

Example 18.—Referring to Example 16, find the present worth of this annuity of $2,500 at 5% if deferred for 5 years and then requiring 10 annuity payments.

Answer.—The entire time is 15 years but the annuity does not begin until 5 years have expired, leaving 10 instalments to be paid. Following the stated rule, we have the following solution.

$\frac{2500}{(1.05)^{15}} \times \frac{(1.05)^{10}-1}{.05} = \$15{,}125.45$, the answer.

Thus, $2{,}500 \times \left(\frac{(1.05)^{10}-1}{.05}\right) = \$31{,}444.73135$, final value.

And $(1.05)^{15} = 2.07892818$, the amount of $1.

Then $\frac{31{,}444.73135}{2.07892818} = \$15{,}125.45$, the answer.

After determining the amount of the annuity for 10 years to be $31,444.73, it is then necessary as shown above to find what sum placed at compound interest will amount thereto in 15 years at the given rate.

Other Similar Problems

If it is desired to determine the present worth of an annuity due or payable at the beginning of the year, proceed in accordance with explanations given under Example 12.

The periods for both annuities and interest may be more frequent than yearly, or the annuity may be yearly and the interest periods half yearly or quarterly. Where the interest is taken half yearly and the instalments yearly, the nominal rate must be changed to the effective yearly rate as per Example 11.

PART VI—DEBT PAID BY EQUAL ANNUAL INSTALMENTS

The example which follows shows the manner of discharging a given debt within a given period by equal annual instalments of both principal and interest. Under this plan each instalment contains interest on all of the bonds then outstanding and part of the principal itself, the latter being either one of the bonds or a portion of the debt. The first year's payment will obviously be largely interest and very little principal, the second less interest and more principal, and so on, during the entire currency of the debt. Each year the interest decreases and the principal increases until the final year which is mostly principal and very little interest. This is an annoying plan of paying debts since the average investor is not able to compute the instalments, and yet it is one that has been more or less popular in England and in Canada. This kind of problem is quite difficult and yet it can if desired be solved by use of the compound interest tables and without the annuity tables or algebra.

Rule.—To determine the annuity for an amount that is payable in equal annual instalments of principal and interest, find the final value thereof for the required time, then divide this final value by the final value of an annuity of $1 for the given time at the given rate. Or, divide the given amount by the present worth of $1 annuity for the required time and rate.

Example 19.—A municipality borrowed $40,000 at 5% interest to be repaid in 15 years by equal annual instalments, including principal and interest. What is the amount of the annual payment? Show the respective amounts paid for principal and interest for the first three years.

Answer.—The same amount must be appropriated each year for 15 years. In other words, what equal annual payment is equivalent to $40,000 invested at 5%—or what annuity can be bought for $40,000 when money is worth 5% per annum. The annuity tables may be readily adapted to the solution of the problem, or the compound interest table itself. The solution may be expressed in the following formula:

Compound amount of $1 $= (1.05)^{15} = \$2.078928$

Compound amount of $40,000 $= 2.078928 \times 40{,}000 = \$83{,}157.12$.

Amount of $1 annuity $= \frac{(1.05)^{15}-1}{.05} = \$21.57856$

Answer $= \frac{83{,}157.12}{21.57856} = \$3{,}853.69$

The aggregate amount of bonds issued is $40,000 while the interest accrued thereon at end of first year is $2,000.

The annual payment however is $3,853.69. This includes the first bond and $2,000 for interest on the entire bond issue; therefore deduct $2,000 from $3,853.69 and we have the amount of the first bond, $1,853.69.

Then 40,000—$1,853.69 = $38,146.31, aggregate of the remaining bonds, which will draw interest for the second year.

$38,146.31 × .05 = $1,907.32 = interest accrued at end of second year.

$3,853.69 = sum of second bond and the interest coupon of $1,907.32.

Deduct $1,907.32 from $3,853.69 and we have the second bond = $1,946.37.

$38,146.31—$1.946.37 = $36,199.94, aggregate of remaining bonds.

The same process may be continued from year to year in determining the part of the annual instalments belonging to principal and the part to interest. It will be noted, however, that in each succeeding year the interest is reduced while the payment on principal is increased.

To illustrate more graphically the working of the annuity bond and the distribution of the annual instalments up to the fourth year, the following table is given:

$40,000 FIRST MORTGAGE, 5%, PAYABLE 15 ANNUAL INSTALMENTS

Equal Annual Payments, $3,853.69

Year	Annual Instalment	Interest	Principal	Remaining Principal
1	$3,853.69	$2,000.00	$1,853.69	$38,146.31
2	3,853.69	1,907.32	1.946.37	36,199.94
3	3,853.69	1,810.00	2,043.69	34,156.25
4	3,853.69	1,707.81	2,145.88	32,010.37

In the books of account, the yearly interest payments will of course be charged to Interest account and the remainder of the annuity against the principal.

From Pennsylvania C. P. A. Examination

An interesting annuity problem from the Pennsylvania Examination is shown below, with solution.

Example 20.—"A manufacturer owes $100,000 on his plant at 5% per annum, due at the end of 5 years from date. He secures an agreement, however, to pay the debt in equal annual instalments which will include principal and interest. What amount is he required to pay each year?"

Answer.—Without further explanation of the principles involved, we shall pass at once to the solution by use of the compound interest table.

Thus, $(1.05)^5$ = $1.27628156, compound amount of $1.

Then $1.27628156 × $100,000 = $127,628.156, amount of principal and interest.

$$\frac{1.27628156-1}{.05} = 5.52563125, \text{ amount of annuity.}$$

Then $127,628.156 ÷ 5,52563125 = $23,097.48, answer.

Or, by following the second rule, divide this present worth of $1

annuity into the given sum to get the required answer. Its application is as follows:

$$\frac{5.52563125}{1.27628156} = \$4.32947667, \text{ the present worth of \$1 annuity.}$$

$$\frac{\$100,000}{4.32947667} = \$23,097.48, \text{ the annual payment required.}$$

The answer is $23,097.48, the required annual payment. An analysis of the interest and principal payments is shown in the following table:

TABLE OF EQUAL ANNUAL INSTALMENTS

Principal, $100,000 payable in 5 years at 5%

Year	Annual Instalment	Interest	Principal	Remaining Principal
1	$23,097.48	$5,000.00	$18,097.48	$81,902.52
2	23,097.48	4,095.13	19,002.35	62,900.17
3	23,097.48	3,145.01	19,952.47	42,947.70
4	23,097.48	2,147.38	20,950.10	21,997.60
5	23,097.48	1,099.88	21,997.60	nil
	$115,487.40	$15,487.40	$100,000.00	

Serial Bond Issues Today

Serial bonds of this nature are difficult for the layman to comprehend because of the complicated calculations. Consequently banking houses usually have the bond instalments represented by even amounts. For example, in a recent serial bond issue of $6,000,000, the bonds begin to mature four years after the date of issue and continue to be paid thereafter in half-yearly instalments. Each semi-annual payment by the company to the trustee is for $270,000, or approximately that amount, which includes interest on all outstanding bonds and a part of the principal. The first instalments of the bonds in question are for $75,000, 77,500, $80,000, $82,500 and so on down the line, increasing each time by $2,500 or more; while the corresponding interest payments are for $195,000, $192,562.50, $190,043.75, $187,443.75, and so on. This simple plan of dividing the payments does away with the compound interest calculations.

PART VII—BOND INVESTMENTS

Bond Premium and Discount

Bonds sell either at par, above par, or below par, depending upon the amount issued, the condition of the money market, the stability of the issuing company, and so forth. If for example a 6% bond sells at par, it is evident that the rate of interest to be paid is exactly 6%. If on the other hand the bond sells above or below par, it is likewise evident that the rate of interest is lower or higher than 6% according to the sale price. In case the bond sells above par, it is evident that the rate of interest in the long run will be less than 6%; but if at a discount, it is likewise evident the rate will be greater than 6%. It is apparent in the case of bonds issued at 90 flat, that while the nominal rate of interest for issuing company is 6% the effective rate is considerably higher. The

company must pay $6.00 interest on every $90 received which is seen to be an even $6\frac{2}{3}\%$; but in addition to this, the company must at maturity pay $10 more than it received. This discount plus the interest is the amount paid for the use of $90.

On the other hand the purchaser of the 6% bonds will net a return upon his investment in proportion to the price paid. In any case, the "yield to maturity" will consist of the annual $6 for interest and a portion of the discount, but this rate of yield is a difficult quotation to determine. The extended bond tables show the exact per cent. of yield for bonds purchased at various prices.

The following quotations respecting discount and premium are taken from Montgomery's Auditing.

Discount on Bonds

"When bonds are sold at a discount it is because the rate of interest the bonds bear is less than the effective rate at which the corporation's credit is rated. For instance, if 5 ten-year bonds are sold at 90, it means that the corporation's borrowing strength is rated at about 6%, and in order to reflect the actual rate each year as interest is paid, it will be necessary to carry the discount as a deferred charge among the assets and write off to interest account 1% annually; this, added to the amount paid in cash, will adjust the interest account to the proper cost."

Premium on bonds

"Where bonds are sold at a premium, the amount received in excess of the par value represents the equivalent of interest collected in advance, and must be held in reserve and distributed over the years to which it applies as a reduction in bond interest account. For instance, a corporation may sell its 5% ten-year bonds at 105, indicating that its credit is rated on a basis of about $4\frac{1}{2}\%$, that is, if a $4\frac{1}{2}$ bond had been issued, the corporation should have realized about par. Therefore the bond interest, when paid, is subject to a deduction of $\frac{1}{2}$ of 1% annually. The excess received at the time of sale should not be applied to income or to surplus, but, as stated above, must be carried as a deferred credit and reduced annually."

Interest Yield on Investments

Investors in bonds frequently purchase them either above or below the par value, depending of course upon the time they have yet to run and the rate of interest which they will draw. Bonds purchased above par manifestly return a lower rate of interest than the coupons call for, and conversely, bonds purchased below par return a higher rate. In determining the true return the rate received by an investor necessitates the amortization of such premium or discount over the life of the bond, and the required portion is either added to or deducted from the cash income at each interest period. In order to illustrate the principles involved in determining the effective rate in amortizing the discount or premium, a few illustrations are given below:

Example 21.—A school district issues bonds for $12,000, with 5 years to run, bearing 6% interest payable yearly. For what sum should they sell, money being worth 5% per annum?

Answer.—There are really two questions to be solved. The $12,000 of bonds at the face value must be paid at the end of 5 years; and the $720 interest must be paid each year for 5 years. It is therefore necessary to find the present worth at 5% of $12,000 due in 5 years, and

the present worth of an annuity of $720 for 5 years, as follows:

Thus, $(1.05)^5 = 1.27628156$, final value of $1

And $12{,}000 \div 1.2762815 = \$9{,}402.314$, the P. W. of $12,000

Then $\frac{(1.05)^5-1}{.05} = 5.52563125$, the final value of $1 annuity

And $\$720 \times \frac{5.52563125}{1.27628156} = \$3{,}117.223$, the P. W. of $720 annuity

Then $9, 402.314 plus $3,117.223 = $12,519.54, the answer

It may be expressed in the following formulas:—

$$\frac{12{,}000}{(1.05)^5} \ \& \ \left(720 \times \left(\frac{(1.05)^5-1}{.05}\right) \div (1.05)^5\right) = \$12{,}519.24$$

$$\text{Or, } 12{,}000 \times \frac{1}{(1.05)^5} \ \& \ \frac{720}{.05}\left(1 - \frac{1}{(1.05)^5}\right) = \$12{,}519.54$$

The bonds should sell for $12,519.54.

Rule.—To determine the cost price of a bond that will return a given rate of interest, the time, rate of interest and rate of yield being given.

Find the present worth of the bond for the given time at the given rate of yield, and to this add the present worth of the interest annuities for the given time at the given rate of yield. The result will be the cost price.

Example 22.—What should be paid for a $1,000 coupon bond maturing five years hence and bearing 4% interest payable annually, so that the investor will receive 5% per annum compound interest on his money?

Answer.—$956.71. The solution to this problem may be stated by the principle followed in the preceding example.

P. W. of $1 due in 5 years at $5\% = \frac{1}{(1.05)^5} = \$.78352617$

P. W. of $1 annuity for 5 years at $5\% = \frac{(1.05)^5}{.05} - 1 \div (1.05)^5 =$ $4.32947667,

Then the P. W. of $1{,}000 = 1{,}000 \times .78352617 = \$783.526$

And the P. W. of the $40 annuity $= 40 \times 4.32947667 = \$173.179$,

Then 783.53 plus 173.18 = $956.71 the required cost.

Amortizing Premium and Discount

The examples which follow are used to illustrate the method of handling premium and discount on bonds, and the manner of determining the rate of yield.

Amortization means the general reduction or extinguishment of a debt or of a given sum. The sinking fund or annual payments for the redemption of a bond or mortgage may be cited as an example, or the depreciation of property.

A common instance of amortization however is applied to a bond which has been purchased at a premium and redeemable after a term of years at par. While the bond draws a certain rate of interest, this can not be entirely considered as a profit, inasmuch as the premium must be written off over a term of years and part of said premium charged against the annual or semi-annual income. The total income or interest received less the amount deducted for amortization, is the net income. The annual income is termed the *nominal* interest; the balance after provision has been made for reduction of premium, is the *effective* interest or true yield of the investment.

Suppose $105 to have been paid for a $100 bond, redeemable in ten years, bearing interest at 5 per cent. per annum. The $5 received each year is the nominal interest and estimated on the par value of the bond, but the true or effective interest yield is somewhat lower and can be determined only after the correct portion of premium has been deducted.

The term "amortization" has come into wide use and is now being used systematically where formerly it was disregarded; yet the method of handling it is complicated and is of little use unless one is thoroughly acquainted with its operation. Because of this inability, tables of bond values have been prepared which give the information respecting bonds purchased at different quotations and bearing different rates of interest. These tables can be purchased or can be seen at almost any banking house.

Determining the Rate of Yield

For bookkeeping purposes, it is necessary to know the rate of yield on the various bond investments. Therefore, the first thing to determine is the true rate of interest after the deduction of premium. This may be had from the bond tables or estimated by means of the compound interest tables. By inspection one may determine very closely the rate of yield that the bond will give, but to prove the accuracy of this assumed rate, it is necessary by successive trials to

(1) Find the present worth of the face of the bond, and
(2) The present worth of the annual or semi-annual interest instalments.

The amount of these two results, if computed at the correct rate of yield, will equal the cost of the bond.

To determine the purchase price of a bond yielding a given nominal rate in order to realize a given return on the outlay, the same method is followed as shown in the preceding illustrations, Examples 21 and 22, in which the necessity of successive trials does not obtain. If the bond is purchased at a discount instead of at a premium, the same principle is used as in the case of premium, but the true yield in such a case would be higher than the nominal rate of interest.

Example 23.—You purchased January 1, 1920, a $10,000 6 per cent. bond having three years to run, for $10,275. If the coupons are payable semi-annually, what per cent. did you make on your investment? Show the entries involved, and the records for bond interest and amortization for the entire time.

Answer.—An answer sufficient for ordinary purposes may be obtained by dividing the bond premium by the number of years the bonds have to run, or better still, to agree with income dates by the number of half years they have to run. The half-yearly income is $300, and one-sixth of the premium is $45.83, which deducted leaves a net income of $254.17, for each period. On this basis, the net annual income would seem to be considerably over 5 per cent. This is not just true, as will be seen below, but it serves as a guide in estimating the rate of yield and in many cases might be considered near enough. Compound interest was not considered, nor interest on the premium, of the use of which the investor has been deprived, therefore the plan must be disregarded.

Now consider the matter from a scientific standpoint. The experimental method used below in determining the true rate of yield may be criticized by mathematicians, but it is no doubt more useful for ordinary

purposes than difficult algebraic formulas. The answer given agrees with the bond tables. By consulting the bond tables, the yield on any bond bought at any given price, can readily be determined. That is why the tables are most useful to bankers and investors

In answering the question, we first determine what rate of yield will be realized on a 6% bond due in three years, bought for a given amount. The rate of interest desired is that rate at which the present worth of the interest and of the principal added together will amount to $10,275 on each bond. To arrive at the correct rate necessitates certain experimental trials. Let us try 5½% and make use of the compound interest tables:

This 5½% will mean six half-years at 2¾% interest
Present worth of $1 annuity for 6 years at 2¾% = $5.46236678
Present worth of $1 compounded for 6 years at 2¾% = .84978491
Then on this basis, the
Present worth of $300 income annuity = 300 × 5.46236678 = $1,638.71
And the present worth of $10,000 compounded = 10,000 × .84978491 = 8,497.85
Present worth of both the bond and interest = $10,136.56

We can see by the above that the annual rate of interest is too high, since it reduces the present worth of the bond and interest instalment to $10,136.56, while the actual present worth or cost is $10,275.00. Then we must try a lower rate of interest and keep on trying until a rate is obtained that will give a present worth of $10,275, or near enough to it for practical purposes.

Let us try 5% this time, or 2½% on half-yearly intervals
*Present worth of $1 annuity for 6 years at 2½% = $5.50812536
*Present worth of $1 compounded for 6 years at 2½% = .86229687
Then on this basis, the
Present worth of $300 annuity = 5.50812536 times 300 = $ 1,652.44
And the Present worth of $10,000 compounded = .86229687 times 10,000 = 8,622.96
Present worth of bond and interest = $10,275.40

*Amounts taken from the annuity tables, but they may be determined from compound interest tables.

We see that the above is correct and that the result is within 40 cents of the cost of the bond; this 40 cents may be spread over the life of the bond or else adjusted either in the opening or closing year.

The Book Records

After determining the rate of yield on the bond the next step is to get the amount to be charged off each year. The entries thereafter are not unusual. When the bond is purchased, debit Bond account for the cost thereof. This will include the premium, of course. At the end of each year the Bond account should be credited with the amount of premium charged off. The balance shows the new amount to be carried forward to the following year, which in turn will be credited at the end thereof with the amount to be written off. The process is continued from year to year, and when the bond matures it will have been reduced by amortization to the par value which is then paid in cash. Another method is to charge a premium account with the amount paid in excess of the par value of the bond. From year to year this account should be credited with the amount of amortization and the balance carried forward to the

next year. By this method the premium is gradually written off over the term of years which constitutes the currency of the bonds.

If the bond is purchased at a discount, debit the bond account for the cost as explained above. At the end of each year the Bond account should be debited with the amount of amortization and thereby gradually increased to the par value thereof. In that case the entries for amortization shown in the illustration would be reversed. It is unwise to write up too freely unless there is a certainty of the bond's being paid in full at maturity.

The book entries in journal form and without providing for monthly accruals are as follows:

January 1, 1920		
Bond Investment	$10,275.00	
To Cash		$10,275.00
For purchase of $10,000 6 per cent. bond due in three years.		
July 1, 1920		
Cash	300.00	
To Income		300.00
For interest coupon on bond investment		
Income	43.12	
To Bond Investment		43.12
For portion of bond premium written off by amortization.		
January 1, 1921		
Cash	300.00	
To Income		300.00
For interest coupons.		
Income	44.20	
To Bond Investment		44.20
Amount of amortization charged to premium.		
July 1, 1921		
Cash	300.00	
To Income		300.00
Income	45.31	
To Bond Investment		45.31
January 1, 1922		
Cash	300.00	
To Income		300.00
Income	46.44	
To Bond Investment		46.44
July 1, 1922		
Cash	300.00	
To Income		300.00
Income	47.60	
To Bond Investment		47.60
January 1, 1923		
Cash	300.00	
To Income		300.00

Income	48.33	
To Bond Investment		48.33
Cash	10,000.00	
To Bond Investment		10,000.00
For payment of 6 per cent. bond maturing this date.		

The Ledger Accounts

Bond Investment Account

(Alien Gas Company 6% Bond due January 1, 1923, To net 5% Interest)

1920			1920		
Jan. 1	To Cash	$10,275.00	July 1	By Income	$ 43.12
			1921		
			*Jan. 1	By Income	44.20
			July 1	By Income	45.31
			1922		
			Jan. 1	By Income	46.44
			July 1	By Income	47.60
			1923		
			Jan. 1	By Income	48.33
			" 1	By Cash	10,000.00
		$10,275.00			$10,275.00

*By right income adjustment should be made on December 31st

Note.—If desired, the account may be balanced at the end of each period and the balance brought down.

Income Account

1920			1920		
July 1	To Bond Investment	$ 43.12	July 1	By Cash	$300.00
" 1	To Profit and Loss	256.88			
1921			1921		
Jan. 1	To Bond Investment	44.20	Jan 1	By Cash	300.00
" 1	To Profit and Loss	255.80			
July 1	To Bond Investment	45.31	July 1	By Cash	300.00
" 1	To Profit and Loss	254.69			
1922			1922		
Jan. 1	To Bond Investment	46.44	Jan. 1	By Cash	300.00
" 1	To Profit and Loss	253.56			
July 1	To Bond Investment	47.60	July 1	By Cash	300.00
" 1	To Profit and Loss	252.40			
1923			1923		
Jan. 1	To Bond Investment	48.33	Jan 1	By Cash	300.00
" 1	To Profit and Loss	251.67			

In case the premium is kept in a separate account, and only the par value entered in the bond account, then the two accounts would appear about as follows:

Bond Investment

(6% Bond cost $10,275, to net 5%)

1920				1920			
Jan. 1		To Cash......	$10,000.00				$10,000.00

Bond Premium

(On $10,000 6% Bond Investment of 1923)

1920				1920			
Jan 1		To Cash.........	$275.00	July 1		By Income....	$ 43.12
				Dec. 31		By Income....	44.20
				1921			
				July 1		By Income....	45.31
				Dec. 31		By Income....	46.44
				1922			
				July 1		By Income...	47.60
				Dec. 31		By Income....	48.33
			$275.00				$275.00

Bonds Purchased Below Par

Example 24.—To illustrate the manner of amortizing the discount on bonds purchased below par, we will assume that a $10,000 bond having five years to run was purchased on January 1, 1920 for $9,573.25 Assuming that the company is thoroughly reliable and that the bond will be paid in full at maturity the accounts for investment and income would be about as follows:

Bond Investment Account

(Central Electric Company 5% of 1925, Yielding 6%)

1920			1925		
Jan. 1	To Cash......	$ 9,573.25	Jan. 1	By Cash.....	$10,000.00
July 1	Amortization..	37.20			
1921					
Jan. 1	Amortization..	38.31			
July 1	Amortization..	39.46			
1922					
Jan. 1	Amortization..	40.65			
July 1	Amortization..	41.87			
1923					
Jan. 1	Amortization..	43.12			
July 1	Amortization..	44.42			
1924					
Jan. 1	Amortization..	45.75			
July 1	Amortization..	47.12			
1925					
Jan. 1	Amortization..	48.85			
		$10,000.00			$10,000.00

Income from Bond Investment

1920			1920		
July 1	To Profit and Loss	$287.20	July 1	By Cash.......	$250.00
				Amortization...	37.20
1921			1921		
Jan. 1	To Profit and Loss	288.31	Jan. 1	By Cash........	250.00
				Amortization...	38.31
July 1	To Profit and Loss	289.46	July 1	By Cash........	250.00
				Amortization...	39.46
1922			1922		
Jan. 1	To Profit and Loss	290.65	Jan. 1	By Cash........	250.00
				Amortization...	40.65
July 1	To Profit and Loss	291.87	July 1	By Cash........	250.00
				Amortization...	41.87
1923			1923		
Jan. 1	To Profit and Loss	293.12	Jan. 1	By Cash........	250.00
				Amortization...	43.12
July 1	To Profit and Loss	294.42	July 1	By Cash........	250.00
				Amortization...	44.42
1924			1924		
Jan. 1	To Profit and Loss	295.75	Jan. 1	By Cash........	250.00
				Amortization...	45.75
July 1	To Profit and Loss	297.12	July 1	By Cash........	250.00
				Amortization...	47.12
1925			1925		
Jan. 1	To Profit and Loss	298.85	Jan. 1	By Cash........	250.00
				Amortization...	48.85

Note.—It is more than likely that many of the above entries would bear dates of June 30th and December 31st, instead of July 1st and January 1st.

Another Method of Solving

Here is another method of working out Examples 23 and 24 that has been presented by one of my students in Accountancy, Mr. D. J. McRae. It is of interest because of the way in which the correct rate of interest is determined. The method is applied to both of the problems, as follows:

Second Solution of Example 23

There is a premium of $275 ($2\frac{3}{4}$% of the par value) to be spread over a period of six half-years at 3% (6% per annum). The problem, therefore, is to find what annuity will amount to, or have a final value of $275.00 in six years at 3%.

$275 = final value of unknown quantity

6.46841 = final value of $1.00 annuity in 6 years at 3%

$$\frac{275}{6.46841} = 42.51 \text{ unknown quantity.}$$

$42.51 is therefore the amount to be deducted each period from the income from the bonds. For so short a period I call this $42.50, as that will be near enough for practical purposes. Not allowing for interest

accumulating this will amount in six periods to $255.00. During these six periods the nominal income from the bonds will amount to $1,800 ($300 each six months). The $1800 minus $255 = $1,545.00 = net income for 3 years. $1,545 ÷ 3 = $515 = net income for 1 year ($1,545 is approximately correct) $\frac{\$515}{10275} = .05 = 5\%$ the effective rate of income.

5% is not exact, but it is so close as to bring a discrepancy of only 46 cents at the end of three years. I then use the following scientific amortization of the premium, using 5% as the effective rate:

Proof of Effective Rate (5%)

Periods ½ Year	Interest: Nominal 6% on Par	Interest: Effective 5% on Cost	Bonds: Amortization of Cost to Par	Bonds: Dr.	Bonds: Cr.	Bonds: Balance
1	300.00	256.88	43.12	10275.00	43.12	10231.88
2	300.00	255.80	44.20		44.20	10187.68
3	300.00	254.69	45.31		45.31	10142.37
4	300.00	253.56	46.44		46.44	10095.93
5	300.00	252.40	47.60		47.60	10048.33
6	300.00	251.67	48.33		48.33	10000.00
					10,000.00	
	1800.00	1525.00	270.00	10275.00	10275.00	

In the above, the effective interest is the interest on the last balance (or inventory value) of the bond, taken at the effective rate. The amortization instalment is the difference between the nominal and true or effective interest. Each successive instalment is deducted from the preceding balance, and a new principal thus obtained.

Second Solution, Example 24

Finding the Rate

Amortization of Interest

Period ½ Year	Amortization of Discount	Interest Payments	Deferred Balance of Bond Discount	Semi-Annual Charges to Profit and Loss
			426.75	10.67
1	42.68	250.00	384.07	9.60
2	42.67	250.00	341.40	8.54
3	42.68	250.00	298.72	7.47
4	42.67	250.00	256.05	6.40
5	42.68	250.00	213.37	5.33
6	42.67	250.00	170.70	4.27
7	42.68	250.00	128.02	3.20
8	42.67	250.00	85.35	2.13
9	42.68	250.00	42.67	1.07
10	42.67	250.00		
Total	426.75	2,500.00		58.68

$2,500.	income from interest	
426.75	income from discount	
2,926.75	total income	
58.68	loss from deferred discount	
2,868.07	net income for five years	

2,868.07 ÷ 5 = 573.61 net income for 1 year.

573.61 ÷ 9,573.25 = .0598 = 5.98% effective rate.

For practical purposes call this 6%. Then we have below:

Amortization of Bond Discount

Period ½ Year	Interest: Nominal 5% on Par	Interest: Effective 6% on Cost	Interest: Amortization to Par	Bonds: Dr.	Bonds: Cr.	Bonds: Balance
				9,573.25		
1	250	287.20	37.20	37.20		9,610.45
2	250	288.31	38.31	38.31		9,648.76
3	250	289.46	39.46	39.46		9,688.22
4	250	290.65	40.65	40.65		9,728.87
5	250	291.87	41.87	41.87		9,770.74
6	250	293.12	43.12	43.12		9,813.86
7	250	294.42	44.42	44.42		9,858.28
8	250	295.75	45.75	45.75		9,904.03
9	250	297.12	47.12	47.12		9,951.15
10	250	298.85	48.85	48.85	10,000.00	
Total	2,500.00	2,926.75	426.75	10,000.00	10,000.00	

Depreciation by Compound Interest

There are two or three methods of determining depreciation by compound interest calculations. In figuring depreciation, the object is to provide an annual charge-off which at the end of a given time will equal the cost of the property either with or without a final scrap value.

If for example a machine cost $2,000 and has an estimated life of 10 years and a scrap value of $200, the main consideration is to provide a plan that will wipe off the remaining $1,800 during the 10 years. There are several methods, each having its advantages and disadvantages. The two methods requiring the use of compound interest, neither one of which is used very much in practice, are known as the *sinking fund method* and the *annuity method.*

Under the Sinking Fund Method the procedure is the same as for creating a sinking fund for the redemption of a debt. This plan presupposes an annual withdrawal of cash to be invested, for the creation of a fund which together with the interest accumulations will equal the sum to be replaced. A depreciation reserve must also be created by regular charges to profit and loss. If it is thought inadvisable to withdraw cash for the creation of a sinking fund the depreciation reserve may nevertheless be accumulated on the sinking fund plan. But in that event there will be no income additions and the discrepancy will have to be made up out of profits.

The sinking fund plan while having merit is seldom used, and in Mr. Montgomery's book on tax procedure he says that "this method is in practice seldom followed."

Example 25.—A machine, costing \$4,000 and having 4 years to run, is depreciated on the sinking fund plan. How much should be charged off each year, if money is worth 6% per annum? Show the book entries.

Answer.—Determine the amount which set aside annually and invested at 6% interest will amount to \$4,000 in 4 years. By use of the compound interest tables, find the amount of an annuity of \$1 for 4 years at 6%, then divide this amount into the \$4,000 to get the required appropriation.

Then, $4000 \div \frac{(1.06^4-1)}{.06} = \$914.37$ the required sum

We see that $\frac{(1.06)^4-1}{.06} = 4.374616$, the amount of an annuity of \$1

Hence, $\frac{4,000}{4.374616} = 914.36596$, the result.

Four instalments of \$914.37 must be made though the last will not draw interest. But if no separate cash investment is made the four appropriations will amount to only \$3,657.48, and will therefore be short the required \$342.52 for interest accumulations. In that case, the interest must be taken out of profits, as shown in the following entries.

The Book Entries under the sinking fund method, omitting the consideration of cash investment, is as follows:

(1) Profit and Loss		\$ 914.37	
To Depreciation Reserve			\$ 914.37
First appropriation for depreciation, on the sinking fund plan.			
(2) Profit and Loss		969.23	
To Depreciation Reserve			969.23
Second appropriation	\$ 914.37		
Interest 6% for one year on \$914.37	54.86		
Total	\$ 969.23		
(3) Profit and Loss		1,027.39	
To Depreciation Reserve			1,027.39
Third appropriation	914.37		
Interest one year on \$1,883.60	113.02		
Total	1,027.39		
(4) Profit and Loss		1,089.01	
To Depreciation Reserve			1,089.01
Fourth appropriation	914.37		
Interest one year on \$2,910.99	174.64		
Total	1,089.01		
(5) Depreciation Reserve		4,000.00	
To Machine Account			4,000.00
To write off the machine account.			

If in the above example a sinking fund were actually created, the interest earnings would be charged to the "sinking fund" and not to Profit and Loss. The four annuity instalments of \$914.37 each would be charged to profit and loss and credited to the reserve. The corresponding cash deposits would likewise be made to the sinking fund and the interest accumulations added thereto. Each year as the interest is added a corresponding credit of like amount is made to depreciation reserve. The four instalments plus the interest accumulations will equal the required \$4,000.

The Annuity Method gradually diminishes the investment from the given amount down to nothing, as outlined under Examples 19 and 20.

Example 26.—Use the same problem as in Example 25 but work it out on the annuity plan.

Answer.—We see that \$4,000 is to be wiped off in 4 years and that compound interest must also be provided for. Thus the \$4,000 investment must be paid in four equal annual instalments including principal and interest.

As in Example 19, we have the following formula:

Thus, $(4000 \times 1.06^4) \div \left(\frac{1.06^4-1}{.06}\right) = \$1,154.37$, the answer.

Then $\frac{4000 \times 1.26247696}{4.374616} = \$1,154.37$, the amount required.

Each instalment is composed of some principal and a full year's interest, as follows:

Principal	\$ 914.37
Interest on \$4,000	240.00
Total	\$1,154.37

The yearly divisions of principal and interest are shown in the following summary:

Year	Annuity	Interest	Principal	Machine Value
(Investment)				\$4,000.00
1	\$1,154.37	\$240.00	\$ 914.37	3,085.63
2	1,154.37	185.14	969.23	2,116.40
3	1,154.37	126.98	1,027.39	1,089.01
4	1,154.37	65.34	1,089.03	None
Total	\$4,617.48	\$617.46	\$4,000.02	

The excess of 2 cents is due to the use of half cents, and may be deducted from the final instalment.

The Book Entries under the annuity plan must use the results shown in the above summary. If the total instalment of \$1,154.37 is charged as depreciation, then the interest in each case must be credited as income. The other and better plan would be to deduct the interest

and charge only the principal instalments, per column 3, as depreciation. The first entry would then be

Profit and Loss	$914.37	
To Machine		$914.37
First charge for depreciation		
Instalment	$1,154.37	
Less Interest	240.00	
Balance	914.37	

Disregarding interest, the remaining entries would be for $969.23, 1,027.39 and $1,089.01. The credits may if desired be made to Depreciation Reserve account and at the end closed into Machinery account.

Practical Annuity Problem

The problem given below appeared sometime ago in the Journal of Accountancy together with an algebraic solution.

Example 27.—"A company on January 1, 1911 contemplates the purchase of the equity in a contract which will yield a net income of $100,000 semi-annually for a period of ten years. From the income received the company desires to pay 3 percent. semi-annually on its investment, and to set aside the balance in a sinking fund which, invested at 3 percent. per annum compounded semi-annually, will produce the amount of the original investment. What are the amounts of the semi-annual sinking fund and of the original investment?"

Answer.—Use the compound interest table as a basis. The income is fixed at $100,000 semi-annually for 10 years. The aggregate income will amount to $2,000,000. Out of this income must be provided a dividend of 6 per cent. on the amount invested, after which the remainder is put out at compound interest in order to provide a sinking fund to replace said investment when the income ceases.

The final value of the sinking fund invested semi-annually at 3% per annum will equal the debt, on which the dividend of 6% per annum is to be taken semi-annually.

First take $1 as the sinking fund instalment and get its final value for

20 periods at .015 ($1\frac{1}{2}$)% $= \dfrac{(1+.015)^{20}-1}{.015}$ or thus:—

Compound amount of $1 for 20 years at .015 = 1.34685501

Amount of $1 annuity for 20 years at .015 $= \dfrac{1.34685501-1}{.015} =$ $23.123667333

Then $23.123667333 = amount of sinking fund, and represents also the amount invested. Then 3% of $23.123667333 is the semi-annual dividend = .69371, to be taken out of income.

Out of every $1.69371 of income received $1.00 is set aside for sinking fund purposes and .69371 paid out in dividends. By using these proportions it is an easy matter to determine what part of the entire $100,000 of income should be set aside for the sinking fund and what part for dividends. Or take it this way—On the basis of each $1 of income the proportions would be .5904198 for sinking fund and .4095802 for dividends, and it is apparent that the total income must be divided in the same proportions.

The sinking fund being the remaining income after paying dividends, it is manifest that we have computed the 3% on the balance of each $1

of income and not on the $1 itself; so that each $1 received would be 169.371% of the sinking fund proportion. Then the $100,000 semi-annual income is 169.371% of the sinking fund payment. In other words each sinking fund instalment is $\frac{100,000}{169371}$ of the income.

Then 100,000 ÷ 169,371% = $59,041.98 the sinking fund instalment each period.

Therefore if the final value of $1 annuity is $23.123667333 then the Final value of an annuity of $59,041.98 = 23.123667333 × 59,041.98 = $1,365,267.10*

Purchase price then	=	$1,365,267.10
Sinking fund instalments	=	59,041.98
Dividend each period	=	40,958.02
Total income instalment	=	100,000.00

*(The cents differ from other answers by using 98 cents instead of 99 cents in the sinking fund instalments.)

Solution by T. Edward Ross, C. P. A.

By referring to tables we find that $1.00 invested at the end of each six months at 3 percent. per annum, compounded semi-annually, will amount at the end of ten years to $23.1236671. For each $1.00 placed in the sinking fund the semi-annual dividend would require 3 percent. of $23.1236671, or $.69371. The amount to be placed semi-annually in the sinking fund would be $\frac{100,000}{169,371}$ of $100,000.00, or $59,041.99. The price to be paid in purchase of the contract would be 23.1236671 × 59,041.99, or $1,365,267.32 plus.

PART VIII—ALGEBRAIC SOLUTIONS

Several of the preceding examples were included in a paper read by the author before the Pennsylvania Institute of Certified Public Accountants and later published in the Journal of Accountancy. Examples 19, 20, 23, and 27 were solved by Mr. Otto A. Spies and contributed to a later issue of the Journal, as follows:

"A perusal of Mr. Bennett's article in the June and July Journal of Accountancy is highly interesting and instructive. However, the solution methods used require the application of bond and compound interest tables, and it appears that the stated problems can be made clearer to the mind of the student and those not having access to elaborate bond tables, by the formation of equations, the solution of which should be an easy matter to those familiar with algebra and logarithms."

(19) A municipality borrowed $40,000.00 at 5% interest, for local improvements, to be repaid in 15 years, by equal annual instalments including principal and interest. What is the amount of the annual payment? Show the respective amounts paid for principal and interest for the first three years.

Solution

The accumulated amount of the debt of $40,000.00 borrowed at 5% annual interest, compounded for a period of 15 years, must have, according to the provisions of the problem, the same value as the accumulated

amounts of a series of 15 equal annual payments, representing principal and interest.

From this condition results the fundamental equation, which may be used for the solution of the problem.

It is evident that the accumulated amounts of the series of 15 equal annual payments must all have the same geometrical ratio of increase, so that, if the annual equal payment be designated by X, and the interest be computed at 5% per annum, the sum of the accumulated amounts of the equal payments will be stated in equation form as follows:

1. $\text{Sum} = X(1+1.05+1.05^2+ + + +1.05^{13}+1.05^{14})$

The amounts within parentheses can be greatly contracted as follows:

Multiply on both sides with the factor, which results by division of one member of the geometrical progression by the next preceding one. This factor would be 1.05. Designating the sum by S, the following or second equation is produced.

2. $S\ 1.05 = X(1.05+1.05^2+1.05^3+ + +1.05^{14}+1.05^{15})$

Deducting equation 1 from equation 2, and again expressing the first equation as follows:

1. $S = X(1+1.05+1.05^2+ + + +1.05^{13}+1.05^{14})$

The result is

$S(1.05-1) = X(1.05^{15}-1)$ or

3. $\text{Sum} = X\dfrac{(1.05^{15}-1)}{0.05}$

The fundamental equation for the solution of the given problem is now expressed:

Accumulated amount of \$40,000.00 at 5% annual interest compounded for 15 years, equals: sum of accumulated amounts of annual equal payments, compounded at 5% per annum, or

4. $40{,}000\times1.05^{15} = X\dfrac{(1.05^{15}-1)}{0.05}$ or

$$X = \frac{40{,}000\times1.05^{15}}{\dfrac{(1.05^{15}-1)}{0.05}} = \frac{40{,}000\times1.05^{15}\times0.05}{1.05^{15}-1}$$

or annual equal payment:

5. $X = \dfrac{2{,}000\times1.05^{15}}{1.05^{15}-1}$

Applying logarithms for the computations as follows, the answer is readily reached.

$$\begin{aligned}
\log 1.05 &= 0.0211893\\
15 \log 1.05 &= 0.3178395\\
1.05^{15} &= 2.0789282\\
1.05^{15}-1 &= 1.0789282
\end{aligned}$$

$\log X = \log 2000 + \log 1.05^{15} - \log (1.05^{15}-1)$

$$\begin{aligned}
\log 2000 &= 3.3010300\\
+\log 1.05^{15} &= \underline{0.3178395}\\
& 3.6188695\\
-\log (1.05^{15}-1) &= \underline{0.0329882}\\
\log X & 3.5858813\\
X &= \$3{,}853.70 \text{ answer}
\end{aligned}$$

The respective amounts paid for principal and interest for the first three years, are to be:

	Balance of Principal	Principal Payment	Interest Payment	Total Annual Payment
	\$40,000.00			
End of 1st year...	38,146.30	\$1,853.70	\$2,000.00	\$3,853.70
End of 2d year...	36,199.92	1,946.38	1,907.32	3,853.70
End of 3d year...	34,156.22	2,043.70	1,810.00	3,853.70
End of 4th year...	32,010.33	2,145.89	1,707.81	3,853.70

(20) "A manufacturer owes \$100,000.00 on his plant at 5% per annum, due at the end of five years from date. He secures an agreement, however, to pay the debt in equal instalments, which will include principal and interest. What amount is he required to pay each year?"

1st Solution

The conditions of the problem demand that the sum of all principal payments, which are to be made at the end of the 1st, 2d, 3d, 4th and 5th years, must equal the sum of \$100,000.00.

If the first principal payment to be made at the end of the first year is designated by X_1 and the succeeding principal payments by X_2; X_3; X_4; and X_5, the first fundamental equation is expressed as follows:

1. $X_1+X_2+X_3+X_4+X_5 = \$100{,}000.00$

It is further known that 5% interest is to be paid every year on the unpaid balance of principal, and as the total payments of principal and interest must every year be the same, it follows that the interest payments must decrease, and the principal payments must increase.

It now becomes the question as to how much this increase of principal payment is, in order to satisfy the requirements of the problem.

If the first principal payment is X_1 and the annual interest is 5%, it follows that the:

2d principal payment $= X_2 = 1.05\times X_1$, and the
3d principal payment $= X_3 = 1.05^2\times X_1$, and the
4th principal payment $= X_4 = 1.05^3\times X_1$, and the
5th principal payment $= X_5 = 1.05^4\times X_1$,

As the sum of all five principal payments must of course equal \$100,000.00 the fundamental equation is therefore restated as follows:

$X_1+1.05\ X_1+1.05^2\ X_1+1.05^3\ X_1+1.05^4\ X_1 = \$100{,}000.00$

Or

2. $X_1\ (1+1.05+1.05^2+1.05^3+1.05^4) = \$100{,}000.00$

The geometrical progression according to well-known rules equals

$1+1.05+1.05^2+1.05^3+1.05^4 = \dfrac{1.05^5-1}{0.05}$

This value inserted in equation 2,

$$X_1 = \frac{\$100{,}000.00}{\dfrac{1.05^5-1}{0.05}} = \frac{100{,}000.00\times0.05}{1.05^5-1}$$

Or 3. $X_1 = \dfrac{5000}{1.05^5-1}$

The computation is easiest with logarithms, as follows:

$$\begin{aligned}
\log 1.05 &= 0.0211893\\
5\ \log 1.05 &= 0.1059465\\
1.05^5 &= 1.27628153\\
1.05^5-1 &= 0.27628153\\
\log 5000 &= 3.6989700\\
-\log (1.05^5-1) &= \underline{0.4413521-1}\\
\log X_1 &= 4.2576179
\end{aligned}$$

First principal payment $= X_1 = \$18{,}097.48$

The interest requirement at the end of the first year would be

$\frac{5}{100} \times \$100{,}000.00 = \$5{,}000.00$

and the total payment to be made at the end of the first year,

Principal	\$18,097.48
Interest	5,000.00
Total payment =	\$23,097.48 Answer

The same total payment is to be made, of course, at the end of the 2d, 3d, 4th, and 5th years, and the schedule of payments would be expressed in a graphic manner as follows:

	Principal Payments	Interest Payments	Total Annual Payments
End of 1st year......	\$ 18,097.48	\$ 5,000.00	\$ 23,097.48
End of 2d year.......	19,002.35	4,095.13	23,097.48
End of 3d year.......	19,952.47	3,145.01	23,097.48
End of 4th year......	20,950.10	2,147.38	23,097.48
End of 5th year......	21,997.60	1,099.88	23,097.48
	\$100,000.00	\$15,487.40	\$115,487.40

2d Solution

If it is considered that the debt of \$100,000.00 of the manufacturer, borrowed at the rate of 5% annual interest, is to be liquidated in five equal annual instalments, there would be an accumulated amount at the end of the fifth year, expressed as follows:

Accumulated amount $= 100{,}000 \times 1.05^5$

This amount must equal the accumulated amounts of a series of five annual equal payments, representing principal and interest, compounded at the rate of five percent. per annum. If the annual equal payment is designated by X there would be:

Accumulated amounts of annual payments $= X\ (1.05^4+1.05^3+1.05^2+1.05+1)$.

This value, according to the conditions of the problem, must equal the accumulated amount of the debt at the end of the fifth year, or expressed as an equation:

1. $100{,}000 \times 1.05^5 = X\ (1.05^4+1.05^3+1.05^2+1.05+1)$

or $X = \dfrac{100{,}000 \times 1.05^5}{1.05^4+1.05^3+1.05^2+1.05+1}$

or the value of $1.05^4+1.05^3+1.05^2+1.05+1 = \dfrac{1.05^5-1}{0.05}$ inserted, we have for X the annual equal payment,

$X = \dfrac{100{,}000 \times 1.05^5 \times 0.05}{1.05^5-1}$ or

2. $X = \dfrac{5{,}000 \times 1.05^5}{1.05^5-1}$

By comparing this value with the value of the first principal payment X_1 as computed in the first solution, it will be observed that the first principal payment X_1 is to the combined interest and principal payment X, as $\dfrac{X_1}{X} = \dfrac{1}{1.05^5} = \dfrac{1}{1.27628156}$

or $X = 1.27628156 \times \$18{,}097.48$

or $X = \$23{,}097.48$ Answer

(23) You purchased January 1, 1920, a \$10,000 6 percent. bond having three years to run, for \$10,275. If the coupons are payable semi-annually, what percent. did you make on your investment? Show the entries involved, and the records for bond interest and amortization for the entire time.

Solution

The semi-annual amount of interest of the bond is \$300.00.

The accumulation of all interest payments at a certain unknown rate of interest for six semi-annual periods is:

$300\ (1.0X^5+1.0X^4+ - - - - - - +1.0X+1)$ or

$300\ \dfrac{(1.0X^6-1)}{0.0X}$

The bond of \$10,000 is payable at the end of the sixth period, and the amount of the bond plus the accumulation of all interest payments must equal the purchase price of the bond, invested for a period of 3 years at an unknown rate of interest (X), to be compounded semi-annually.

The fundamental equation for this condition is therefore:

1. $10275 \times 1.0X^6 = \dfrac{300}{0.0X}(1.0X^6-1)+10{,}000.$

or the purchase price:

2. $10275 = \dfrac{300}{0.0X}\left(\dfrac{(1.0X^6-1)}{1.0X^6}\right) + \dfrac{10000}{1.0X^6}$

This equation (2) has only one unknown quantity, X, the desired rate of interest but cannot be solved directly as X appears at the 6th power. The practical way to proceed, in order to compute X, is to take approximate values for X, and insert these values in equation (2).

If such an inserted value for X satisfies the equation (2) perfectly, it must be the exact and mathematically correct amount of the desired rate of interest.

The nominal rate of interest being 3% semi-annually, and the bond premium being \$275.00, it follows that the effective rate of interest must be less than 3%.

An approximate correct amount is ascertained from the approximate return of the bond.

In the given example, there is a nominal periodical return of \$300.00.

If the approximate periodical amortization $\frac{275}{6} = \$45.83$ is deducted from the nominal return, there results the approximate return of the bond of \$10,000.

\$300—\$45.83 = 254.17, or for a \$100.00 bond = \$2.5417.

From this result it may be concluded that the true effective semi-annual interest is close to 2.5%.

If for the purpose of verification, this value for X = 2.5 is inserted in equation (2), there would result:

$$3.\ 10275 = \frac{300}{0.025}\left(\frac{1.025^6-1}{1.025^6}\right) + \frac{10{,}000}{1.025^6} \text{ or}$$

$$10275 = 12000\left(\frac{0.1596934}{1.1596934}\right) + \frac{10000}{1.1596934}$$

4. 10275 = 1652.44 + 8622.96 = 10275.40

or indicating that the semi-annual rate of $2\frac{1}{2}\%$ is within 40 cents mathematically correct and suffices for practical purposes.

The mathematically correct amount for X may be ascertained in the following way:

If in equation (2) X = 2.51 is inserted, the result would be:

5. 10275 = 1651.88 + 8617.93 = 10269.81 whereas if X = 2.50

4. 10275 = 1652.44 + 8622.96 = 10275.40.

From this may be concluded that if X increases by 0.01, the result of the equation decreases by \$5.59. The correct increase of X, in order to satisfy equation (2) perfectly, must be:

$$\text{Increase of } X = \frac{40 \times 0.01}{559} = \frac{40}{55900} = 0.0007$$

The true value of X is therefore X = 2.5 + 0.0007 = 2.5007

If this value is inserted in equation (2) the result is:

$$6.\ 10275 = \frac{300}{0.025007}\left(\frac{1.025007^6-1}{1.025007^6}\right) + \frac{10000}{1.025007^6}$$

or computed with logarithms:

$$\begin{aligned} \log 1.025007 &= 0.0107269 \\ 6 \log 1.025007 &= 0.0643614 \\ 1.025007^6 &= 1.159742 \\ 1.025007^6 - 1 &= 0.159742 \end{aligned}$$

$$10275 = \frac{300}{0.025007}\left(\frac{0.159742}{1.159742}\right) + \frac{10000}{1.159742} = \frac{47.9226}{0.0290016} + \frac{10000}{1159742}$$

10275 = \$1652.40 + 8622.60 = 10275.

This proves that the mathematically correct amount of the semi-annual effective rate of interest is 2.5007%.

PERIODICAL AMORTIZATION BASED ON TRUE OR EFFECTIVE RATE OF INTEREST

Take the correct effective rate of semi-annual interest or the rate of yield on the purchase price of the bond at 2.5007%, and the nominal income, \$300.00.

There is for the first semi-annual period, a true income of 10275 × 0.025007.

True income = \$256.95. The difference between the nominal income and the true income is \$43.05.

This amount of \$43.05 is the first periodical amortization and should be charged at the end of the first period to income and credited to bond investment, if the bond investment were charged with the purchase price.

Dr. Income..................\$43.05
Cr. Bond investment.......... 43.05

Or, if a separate bond premium account is kept:

Dr. Income
Cr. Bond premium account.

For the second period, the bond investment has decreased by \$43.05 and is now 10,275.00—43.05 = 10231.95, on which the true income is 10231.95 × 0.025007 = 255.87, and the amortization \$44.13.

These operations are to be continued semi-annually six times, until at the end of the third year, the entire bond premium is entirely charged off.

The periodical amortization, true income and bond balances, computed on the basis of true effective interest rate for the full period, are as follows:

Period	Nominal Income	True Income	Bond Amortization	Bond Purchase Price	Bond Balance
1	\$300.00	\$256.95	\$43.05	\$10,275.00	\$10,231.95
2	300.00	255.87	44.13	....	10,187.82
3	300.00	254.77	45.23	....	10,142.59
4	300.00	253.64	46.36	....	10,096.23
5	300.00	252.48	47.52	....	10,048.71
6	300.00	251.29	48.71	....	10,000.00
	\$1,800.00	\$1,525.00	\$275.00	\$10,275.00	

Another more scientific way for computing the exact periodical amortization, after the true effective rate of interest has been obtained, is as follows:

If the first periodical amortization is designated by X, the aggregate accumulation for 6 periods at 2.5007% true interest compounded semi-annually must equal the bond premium. Or expressed by equation:

$$7.\ X(1 + 1.025007 + 1.025007^2 + + + 1.025007^5) = \$275.00$$

$$\text{or } X\left(\frac{(1.025007^6-1)}{0.025007}\right) = 275$$

$$\text{or } X = \frac{275 \times 0.025007}{1.025007^6-1}$$

$$X = \frac{6.876925}{1.025007^6-1}$$

Computed with logarithms:

$$\begin{aligned} \log 1.025007 &= 0.0107269 \\ 6 \log 1.025007 &= 0.0643614 \\ 1.025007^6 &= 1.159742 \\ 1.025007^6 - 1 &= 0.159742 \end{aligned}$$

$$\text{or } X = \frac{6.876925}{0.159742} = \$43.05$$

Equation (7) determines the following amortizations as:

1st amortization X		=	$ 43.05
2d	"	43.05×1.025007 =	44.13
3d	"	43.05×1.025007^2 =	45.23
4th	"	43.05×1.025007^3 =	46.36
5th	"	43.05×1.025007^4 =	47.52
6th	"	43.05×1.025007^5 =	48.71
			$275.00

For the purpose of comparing results, there is given a tabulation of the mathematically correct figures and those resulting from approximate methods.

Period	True Income	Exact Amortization	Approximate Amortization	True Bond Balance	Approximate Bond Balance
1st	$256.95	$43.05	$43.12	$10,231.95	$10,231.88
2d	255.87	44.13	44.20	10,187.82	10,187.68
3d	254.77	45.23	45.31	10,142.59	10,142.37
4th	253.64	46.36	46.44	10.096.23	10,095.93
5th	252.48	47.52	47.60	10,048.71	10,048.33
6th	251.29	48.71	48.33*	10,000.00	10,000.00
	$1,525.00	$275.00	$275.00	....	....

* Adjusted by 46 cents from $48.79 to $48.33.

(27) A company on January 1, 1911, contemplates the purchase of the equity in a contract which yields a net income of $100,000.00 semi-annually over a period of ten years.

From the income received, the company desires to pay 3 percent. semi-annually on its investment, and to set aside the balance in a sinking fund which, invested at 3% per annum, compounded semi-annually, will produce the amount of the original investment.

What are the amounts of the semi-annual sinking fund, and of the original investment?

Solution

In this problem are two unknown quantities, namely,

The original investment or purchase price, designated by X,

The amount of the sinking fund, which invested at 3% per annum, compounded semi-annually, will produce the amount of the original investment, designated by Y.

According to the conditions of the problem, the accumulations of the sinking fund contributions, compounded semi-annually at 3% per annum, must equal the original purchase price X, or expressed by an equation:

1. $X = Y\,(1.015^{19} + 1.015^{18} + \text{- - - -} + 1.015 + 1)$

or contracted, according to well-known rules,

2. $X = Y\left(\frac{1.015^{20} - 1}{0.015}\right)$

A certain part of the semi-annual income of $100,000.00 has to be set aside as sinking fund contribution = Y, and the balance of the $100,000.00 represents and must equal 3% semi-annual interest on the purchase price X.

From this condition follows a second fundamental equation:

$100{,}000 = Y + \frac{3X}{100}$ or

3. $Y = 100{,}000 - \frac{3X}{100}$

If this value of Y is inserted in the equation 2, there is obtained a very simple equation of the first degree for X, from which the unknown purchase price can be easily computed, namely:

4. $X = \left(100{,}000 - \frac{3X}{100}\right) \times \left(\frac{(1.015^{20} - 1)}{0.015}\right)$

or dissolved

$$X + \frac{3X}{100}\left(\frac{1.015^{20} - 1}{0.015}\right) = 100{,}000\left(\frac{1.015^{20} - 1}{0.015}\right) \text{ or}$$

5. $\text{Purchase price} = X = \dfrac{100{,}000\left(\frac{1.015^{20} - 1}{0.015}\right)}{1 + \frac{3}{100}\left(\frac{1.015^{20} - 1}{0.015}\right)}$

The value $\frac{(1.015^{20} - 1)}{0.015}$ is to be best computed with logarithms, as follows:

$$\begin{aligned} \log 1.015 &= 0.00646604 \\ 20 \log 1.015 &= 0.1293208 \\ 1.015^{20} &= 1.34685501 \\ 1.015^{20} - 1 &= 0.34685501 \\ \frac{1.015^{20} - 1}{0.015} &= 23.123667333 \end{aligned}$$

This value inserted in equation 5, results in

$$X = \frac{100{,}000 \times 23.123667333}{1 + \frac{3}{100}\,(23.123667333)} \text{ or}$$

6. $X = \frac{2312366.7333}{1.69371002} = \$1{,}365{,}267.19$ Answer.

For computation of the semi-annual sinking fund contribution,we use best equation 3, namely,

3. Sinking fund contribution $= Y = 100{,}000 - \frac{3X}{100}$ or inserting the value of X,

$Y = 100{,}000 - \frac{3 \times 1{,}365{,}267.19}{100}$ or

$Y = 100{,}000 - 40{,}958.0157$

7. $Y = \$59{,}041.98$ Answer.

With these values for X and Y, the conditions of the problem are satisfied in every respect, namely:

2. $X = Y\left(\frac{1.015^{20} - 1}{0.015}\right)$ or values inserted:

$1{,}365{,}267.19 = 59{,}041.98 \times 23.123667333$ and also

3. $Y = 100{,}000 - \frac{3X}{100}$ or values inserted:

$59{,}041.98 = 100{,}000 - \frac{4{,}095{,}801.57}{100}$

COMPOUND INTEREST TABLE

Giving the Amount of $1 at Compound Interest

Years	1 Percent.	2 Percent.	2½ Percent.	3 Percent.	3½ Percent.
1	1.0100 0000	1.0200 0000	1.0250 0000	1.0300 0000	1.0350 0000
2	1.0201 0000	1.0404 0000	1.0506 2500	1.0609 0000	1.0712 2500
3	1.0303 0100	1.0612 0800	1.0768 9062	1.0927 2700	1.1087 1788
4	1.0406 0400	1.0824 3216	1.1038 1289	1.1255 0881	1.1475 2300
5	1.0510 1005	1.1040 8080	1.1314 0821	1.1592 7407	1.1876 8631
6	1.0615 2015	1.1261 6242	1.1596 9342	1.1940 5230	1.2292 5533
7	1.0721 3535	1.1486 8567	1.1886 8575	1.2298 7387	1.2722 7926
8	1.0828 5671	1.1716 5938	1.2184 0290	1.2667 7008	1.3168 0904
9	1.0936 8527	1.1950 9257	1.2488 6297	1.3047 7318	1.3628 9735
10	1.1046 2213	1.2189 9442	1.2800 8454	1.3439 1632	1.4105 9876
11	1.1156 6835	1.2433 7431	1.3120 8666	1.3842 3387	1.4599 6972
12	1.1268 2503	1.2682 4179	1.3448 8882	1.4257 6089	1.5110 6866
13	1.1380 9328	1.2936 0663	1.3785 1104	1.4685 3371	1.5639 5606
14	1.1494 7421	1.3194 7876	1.4129 7382	1.5125 8972	1.6186 9452
15	1.1609 6896	1.3458 6834	1.4482 9817	1.5579 6742	1.6753 4883
16	1.1725 7864	1.3727 8571	1.4845 0562	1.6047 0644	1.7339 8604
17	1.1843 0443	1.4002 4142	1.5216 1826	1.6528 4763	1.7946 7555
18	1.1961 4748	1.4282 4625	1.5596 5872	1.7024 3306	1.8574 8920
19	1.2081 0895	1.4568 1117	1.5986 5019	1.7535 0605	1.9225 0132
20	1.2201 9004	1.4859 4740	1.6386 1644	1.8061 1123	1.9897 8886
21	1.2323 9194	1.5156 6634	1.6795 8185	1.8602 9457	2.0594 3147
22	1.2447 1586	1.5459 7967	1.7215 7140	1.9161 0341	2.1315 1158
23	1.2571 6302	1.5768 9926	1.7646 1068	1.9735 8651	2.2061 1448
24	1.2697 3465	1.6084 3725	1.8087 2595	2.0327 9411	2.2833 2849
25	1.2824 3200	1.6406 0599	1.8539 4410	2.0937 7793	2.3632 4498
26	1.2952 5631	1.6734 1811	1.9002 9270	2.1565 9127	2.4459 5856
27	1.3082 0888	1.7068 8648	1.9478 0002	2.2212 8901	2.5315 6711
28	1.3212 9097	1.7410 2421	1.9964 9502	2.2879 2768	2.6201 7196
29	1.3345 0388	1.7758 4469	2.0464 0739	2.3565 6551	2.7118 7798
30	1.3478 4892	1.8113 6158	2.0975 6758	2.4272 6247	2.8067 9370

NOTE.—To obtain the compound interest for a greater number of years than given on the table, multiply together the amounts for any two number of years that will equal the required number. For example, required amount for 40 years at 3% is obtained by multiplying the compound amount for 20 and 20 years, or by 10 and 30, and so on.

COMPOUND INTEREST TABLE

Giving the Amount of $1 at Compound Interest

Years	4 Percent.	5 Percent.	6 Percent.	7 Percent.	8 Percent.
1	1.0400 0000	1.0500 0000	1.0600 0000	1.0700 000	1.0800 000
2	1.0816 0000	1.1025 0000	1.1236 0000	1.1449 000	1.1664 000
3	1.1248 6400	1.1576 2500	1.1910 1600	1.2250 430	1.2597 120
4	1.1698 5856	1.2155 0625	1.2624 7696	1.3107 960	1.3604 890
5	1.2166 5290	1.2762 8156	1.3382 2558	1.4025 517	1.4693 281
6	1.2653 1902	1.3400 9564	1.4185 1911	1.5007 304	1.5868 743
7	1.3159 3178	1.4071 0042	1.5036 3026	1.6057 815	1.7138 243
8	1.3685 6905	1.4774 5544	1.5938 4807	1.7181 862	1.8509 302
9	1.4233 1181	1.5513 2822	1.6894 7896	1.8384 592	1.9990 046
10	1.4802 4428	1.6288 9463	1.7908 4770	1.9671 514	2.1589 250
11	1.5394 5406	1.7103 3936	1.8982 9856	2.1048 520	2.3316 390
12	1.6010 3222	1.7958 5633	2.0121 9647	2.2521 916	2.5181 701
13	1.6650 7351	1.8856 4914	2.1329 2826	2.4098 450	2.7196 237
14	1.7316 7645	1.9799 3160	2.2609 0396	2.5785 342	2.9371 936
15	1.8009 4351	2.0789 2818	2.3965 5819	2.7590 315	3.1721 691
16	1.8729 8125	2.1828 7459	2.5403 5168	2.9521 638	3.4259 426
17	1.9479 0050	2.2920 1832	2.6927 7279	3.1588 152	3.7000 181
18	2.0258 1652	2.4066 1923	2.8543 3915	3.3799 323	3.9960 195
19	2.1068 4918	2.5269 5020	3.0255 9950	3.6165 275	4.3157 011
20	2.1911 2314	2.6532 9771	3.2071 3547	3.8696 845	4.6609 571
21	2.2787 6807	2.7859 6259	3.3995 6360	4.1405 624	5.0338 337
22	2.3699 1879	2.9252 6072	3.6035 3742	4.4304 017	5.4365 404
23	2.4647 1555	3.0715 2376	3.8197 4966	4.7405 299	5.8714 637
24	2.5633 0417	3.2250 9994	4.0489 3464	5.0723 670	6.3411 807
25	2.6658 3633	3.3863 5494	4.2918 7072	5.4274 326	6.8484 752
26	2.7724 6978	3.5556 7269	4.5493 8296	5.8073 529	7.3963 532
27	2.8833 6858	3.7334 5632	4.8223 4594	6.2138 676	7.9880 615
28	2.9987 0332	3.9201 2914	5.1116 8670	6.6488 384	8.6271 064
29	3.1186 5145	4.1161 3560	5.4183 8790	7.1142 571	9.3172 749
30	3.2433 9751	4.3219 4238	5.7434 9117	7.6122 550	10.0626 569

Review Questions

The following questions should be studied and fully mastered, but they are not to be sent in.

1—Give the definition for interest, discount, compound interest, compound discount. 2—What is included under the name actuarial science? Annuities? 3—You discount at the bank your note for two calendar months at 6%. How much interest will the bank deduct, and what effective rate will this be equivalent to on the amount borrowed? 4—On what plan do banks in your city figure interest—on the 360-day plan or the 365-day plan? 5—What is the theory of increase with respect to compound interest? What is the difference between compound amount and compound interest? 6—What is meant by exponents? Powers? Equations? Formula? Compound amount? 7—What is meant by interpolation as applied to compound interest calculations? 8—What is meant by logarithms? Are you familiar with their use? 9—Give the definition of annuity, an annuity certain, perpetual annuity, annuity in possession, deferred annuity, annuity in arrears. 10—What is the difference between the amount of an annuity and the present worth? Between compound amount of $1 and compound amount of $1 annuity? 11—Why are annuity payments made at the end of the year instead of at the beginning? 12—Why do annuities refer to "periods" and not necessarily to years? What may a period comprise? 13—What is meant by amortization? Mention the different applications thereof. 14—What is a sinking fund, as referred to in this lesson? 15—What is the rule for determining the final value of an annuity? The present worth? The sinking fund amounts? 16—Give an algebraic formula for determining each of the above. 17—What causes the difference between the final value of an "ordinary annuity" and of an "annuity due" or prepaid? 18—How do you find the present value of an annuity in possession? Of a deferred annuity? Of an annuity in reversion? 19—Some bonds are paid in equal annual instalments of principal and interest. Prepare a formula for determining the instalments. 20—What is the object of issuing serial bonds of this nature instead of the regular bonds maturing at a given date? 21—What is contained in a table of compound interest? Table of logarithms? Table of sinking funds? 22—Give an explanation of the bond tables, what information they contain, etc. 23—Have you had an opportunity of examining or studying these tables? 24—What is meant by bond premium? Bond discount? 25—What governs the sale price of a bond, whether a premium or discount? 26—How should a premium be amortized on the books of the issuing company? On the books of the investor? 27—How should discount be amortized on the books of the company? On the books of the investor? 28—How do you find the actual or effective rate of return on a bond purchased at a premium? At a discount? 29—Why are approximations necessary in determining the effective rate of a return? Is this always necessary? 30—Can you submit a formula that will give the correct rate of return? 31—What is meant by the sinking fund method of depreciation? The annuity method? How do these differ? 32—Can you state the rules for working the different examples contained in this lesson? 33—Why are bond tables prepared on the basis of six-month periods? That is, why are they compounded half-yearly instead of yearly? (Because bond

coupons are nearly always payable half yearly). 34—Can you make the book entries for the various examples illustrated in this lesson? 35—The interest return on investments is spoken of as either the "stock yield" or as the "bond yield to maturity." How do these differ? 36—Why does the last payment of an annuity not draw interest?

Review Problems

These problems with answers are given as a means of testing your knowledge of annuities. Prove the answers.

1—What sum deposited in a savings bank paying 4% compound interest will amount to $5,000 in 25 years? Answer, $1,875.58.

2—What will a $2,000 annuity amount to in 5 years if compounded annually at 5%? Answer, $11,051.26.

3—What will a $4,000 annuity due, or prepaid, amount to in 5 years if compounded annually at 5%? Answer, $23,207.65.

4—On July 1, 1922, a company issued $10,000,000 of 5% bonds payable in 20 years. What amount should be set aside annually and compounded at 4½% to meet the debt when due? Answer, $318,761.44.

5—An annuity of $1,200 remained unpaid for 8 years. With interest worth 6%, what was the amount due? Answer, $11,876.96.

6—What is the present worth of a perpetual scholarship that will pay $400 annually, at 5% interest? Answer, $8,000.

7—Find the present worth of an annual pension of $240 for 10 years at 4% interest. Answer, $1,946.61.

8—An annuity of $5,000 is to begin in 6 years and continue for 12 years. Find the present worth thereof at 6%. Answer, $29,551.40.

9—Find the present worth of a perpetual annuity of $1,000 at 5% interest. Answer, $20,000. What would it be if the annuity were deferred 3 years? Answer, $17,276.75.

10—A debt of $120,000 due in 10 years is to be paid in 10 equal annual instalments of principal and interest, with interest compounded annually at 5%. State the annual payment. Answer, $15,540.06.

11—You purchased $4,000 par of 8% bonds due 4 years hence at a price to net 7% on your investment. Find the cost. Answer, $4,137.60.

12—You purchased for $820 a $1,000, 4½% bond due in two years. What rate of interest did you make on your money? Answer, 15%. Now prove it.

13—Bonds bearing 5% interest and due in 10 years were purchased on a 7% basis. Find the cost. Answer, $92.89.

14—A company sold $500,000 of 10-year 5% bonds to the underwriters for $454,050. Considering the bond discount, what rate is the loan costing? Answer, 6¼%.

15—A company owes a debt of $21,000 drawing 6% interest which it has agreed to pay off in semi-annual instalments of $3,000 each How long will it take to pay the debt reckoning interest half-yearly? Answer, 8 half-yearly payments.

EXAMINATION—Lesson 16

Prepare answers to the following questions and send them in for inspection. Submit only your very best efforts, and prove your results before submitting them for approval.

1. Do you clearly understand the matter covered in this lesson? Can you answer all of the review questions? Review problems?

2. Explain carefully each of the following terms:
   (a) Actuarial science.
   (b) Compound discount.
   (c) An annuity due.
   (d) Sinking fund.
   (e) Present value of a deferred annuity.
   (f) Effective interest rate.
   (g) Logarithms.
   (h) Amortization.

3. Explain and illustrate each of the following terms:
   (a) Ratio of increase.
   (b) Compound amount.
   (c) Formula.
   (d) Equation.
   (e) Exponent and power.
   (f) Interpolation.

4. From the compound interest table, construct from the 6% column the following tables up to the fourth year, and carried to six places.
   (a) Amount of an annuity table for 4 years at 6%.
   (b) Amount of annuity due or prepaid for 4 years at 6%.
   (c) Present worth of an annuity table for 4 years at 6%.
   (d) Sinking fund or annuity table for 4 years at 6%, showing amounts that invested at the end of each period will amount to $1.

Give in each case the formulas used and the method of solution.

5. What is the final value of an annuity of $5,000 for four years at 6% if the interest is compounded half-yearly? Give the formulas and analysis.

6. What sum of money deposited at the end of each year for the next four years will amount to $3,000, if money is compounded annually at 4%?

7. An electric light company on January 1, 1922, issued $10,000,000 10-year 6% sinking fund bonds. The sinking fund is to consist of annual deposits with the trustee who in turn will deposit them in a savings bank at 4% interest to be compounded half-yearly. Required, with full details:
   (a) The amount to be desposited annually.
   (b) The book entries and accounts for sinking fund the first 3 years.

8. What is the present value of an annuity of $10,000 having 3 years to run after being deferred 3 years, if interest is worth 5% compounded annually? Give full analysis. Illustrate your answer by a schedule of reductions.

9. You have purchased a mortgage that is payable in three equal annual payments of $4,000 each for principal and interest combined, the first payment to become due one year from now.

Find how much of this is for principal and how much for interest, reckoning interest at 6% per annum, compounded yearly. Make all necessary journal entries on your books.

10. Find the amount to be raised annually at the end of each year for the next 3 years to provide a Sinking Fund to pay off a loan of $100,000 and to pay interest on the loan at 6%, if the rate of interest for the investment of the Sinking Fund is to be taken at 5% per annum compounded annually.

11. A company purchased a piece of real estate for $100,000. The terms of payment agreed upon were $10,000 each year thereafter, without interest, until the whole $100,000 was paid. At the end of the fourth year, twelve months before the fifth payment was due, the company found itself possessed of considerable available cash, and decided to relieve itself of the liability specified by depositing one amount sufficient to pay the annual instalments as they matured. The bank agreed to pay 5% per annum on all money deposited for that purpose.

What is the single amount which the company must deposit, at 5%, to pay the yearly instalments as they mature?

12. A broker offered you $10,000 of 5% bonds, having 3 years to run, interest payable yearly, at a price to yield 6% on the investment.

Find how much you would have to pay for these bonds, and prepare a statement to show that the amounts to be received under the bonds for principal and interest would repay the amount invested, together with interest thereon at 6%, as expected.

13. A lease has five years to run at $1,000.00 a year payable at the end of each year, with an extension for a further five years at $1,200.00 a year. On a 6% basis what sum should be paid now in lieu of the ten years' rent? (V^5 at $6\% = .7473$).

14. You purchased $20,000 of 8% bonds having 30 months yet to run, for $20,532.
   (a) What rate of interest do you realize on your investment, allowing interest to be compounded half-yearly? Show work.
   (b) Show the necessary book accounts and journal entries.

15. You paid $972.90 for a $1,000 5% bond that is to fall due in exactly 3 years. What rate will you realize on the investment? Give full analysis.

16. A corporation desires to retire in five payments a debt of $150,000 bearing 5% interest payable annually. The fifth payment, including interest, is to be $30,000. The other four equal periodical payments are all to include interest. Make a schedule of the five instalments.

17. A machine which cost $10,000 and has a scrap value of $1,000 is to be fully depreciated by the annuity method in 5 years. Using a 6% rate, show the annual depreciation, the entries, and the accounts.

18. A company decides to create a Sinking Fund of $50,000 to replace an asset at the end of 3 years by charging to Depreciation at the end of each year for the next 3 years such a sum as will accumulate with interest

added at 5% per annum, compounded annually, to $50,000 at the end of the 3 years, the money in the meantime being kept in the business and not specially invested, and the interest being provided out of the general revenue of the business and charged to interest account at the end of each year. Find what amounts should be charged annually to Depreciation and Interest accounts respectively, and prepare a statement of the Sinking Fund showing it accumulated to $50,000.

19. A Savings Bank on July 1, 1921, purchased 50 bonds of par value $1,000.00 each of the A. B. R. R. Co. The bonds mature January 1, 1930, and bear interest at the rate of 4½% per annum, payable semi-annually. They were purchased on a semi-annual basis of 2%.

(1) What was the total cost of the bonds?
(2) Construct a Schedule of Amortization for the premium on the bonds.
(3) Set up an account with the bond issue, showing what entries would be made on each interest date.

Note—Most of the above problems are taken from or based upon official examinations.

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Note. Most of the above problems are taken from or based upon official examinations.

GET THE DICTIONARY HABIT

Purchase a pocket dictionary and carry it with you; also purchase a larger one for home use. If you will use them as we direct, they will prove the best investment of your life.

You often hear people use words, and find words in reading, the meaning of which you do not understand. You also sometimes find that you cannot pronounce or spell words that you wish to use in conversation or writing. The proper understanding and use of these words would add greatly to your powers of comprehension and expression.

To have something worth saying, and not be able to say it because **you cannot find the words** with which to express your meaning, or because **you are afraid to use words for fear** you will not pronounce them correctly, is a great handicap. It makes you self-distrustful and self-conscious, and prevents you from getting credit for your full worth among your business or other associates. To know, but be unable to say or write what you know clearly and intelligently, is a great disadvantage, either from a business or a social standpoint.

Every time you meet a word in reading or conversation that you do not understand or cannot pronounce or spell, consult your dictionary. By doing this you will learn the meaning of words and how to pronounce and spell them. You will, in the course of a few years, add greatly to your knowledge and improve you pronunciation and orthography.

The greatest and most famous authors write with a dictionary at their elbow, and refer to it constantly. In no other book is so much information condensed. A good dictionary is a summary of the whole store of human intelligence.

END OF
TITLE